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Economic lot-sizing problem with remanufacturing option: Complexity and Algorithms

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Abstract In a single item dynamic lot-sizing problem, we are given a time horizon and demand for a single item in every time period. The problem seeks a solution that determines how much to produce and carry at each time period, so that we will incur the least amount of production and inventory cost. When the remanufacturing option is included, the input comprises of number of returned products at each time period that can be potentially remanufactured to satisfy the demands, where remanufacturing and inventory costs are applicable. For this problem, we first show that it cannot have a fully polynomial time approximation scheme (FPTAS). We then provide a pseudo-polynomial algorithm to solve the problem and show how this algorithm can be adapted to solve it in polynomial time, when we make certain realistic assumptions on the cost structure.

Keywords Lot-sizing; Remanufacturing; Complexity; Polynomial Algorithms

1 Introduction

The classical lot-sizing problem is defined over a finite planning horizon with discrete time periods. The demand for a single item in each time period is provided as an input. The demand could be satisfied by either manufacturing the item or through the inventory carried from previous period. There are no

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restrictions on how much we can manufacture in a given period. Items produced in excess of demand are carried over to the next period in the inventory. The objective is to determine the least cost production plan that satisfies the demands at every period. With the remanufacturing option, the demands can be satisfied either by manufacturing new items or remanufacturing returned items. The returns at every period are provided as an input to the problem. The problem consists of separate inventory costs for carrying remanufactured items and manufactured items (sometimes referred to as serviceable inventory), and there is also a cost incurred for manufacturing or remanufacturing. Remanufacturing is the process of recovering used products by repairing and replacing worn out components so that a product is created at the same quality level as a newly manufactured product. This saves tonnes of landfill every year by providing an environmentally friendlier alternative to classical manufacturing. It also offers industries the potential to significantly save money by exploiting used product inventories and reusing many precious raw materials that are becoming increasingly scarcer. With this motivation, we study the single item production planning problem over a finite horizon with the option of remanufacturing.

The classical lot-sizing problem was introduced in [16] by Wagner and Whitin, where the manufacturing has an unrestricted capacity. They provided a dynamic program that can solve this problem in polynomial time. Various variants of it have been thoroughly studied over the last 6 decades, see [3] for a recent review. Later, the capacitated version was introduced and the problem was shown to be NP-hard, see [5]. A dynamic program was provided in [4] which runs in polynomial time for unary encoding. A fully polynomial time approximation scheme (FPTAS) was provided in [8]. There are a number of variations to the classical lot sizing problem (see for instance [1, 10]). The most pertinent variation to this study with remanufacturing option was first studied in [7] and proved as NP-hard in [13]. A dynamic program with polynomial running time was provided for a special case of when the cost involved are time invariant and there is a joint set-up cost involved for both manufacturing and remanufacturing [12]. A polynomial time algorithm was provided when all costs are linear by solving it as a flow problem [7]. Since then, very little progress has been made for polynomial special cases. The general variations of the problem have been shown to be NP-hard [11]. In addition several tight formulations and their comparisons based on their lower bounds were provided in [11]. In [9], the authors exploit the optimality structure to decompose the problem into polynomially solveable subproblems. A heuristic procedure was then provided, where a polynomial subset of these subproblems were then chosen and solved.

1.1 Organisation

We first show in section 2 that the general case of this problem cannot have an FPTAS unless P=NP. We refer the reader to [6] for concepts about NP-hardness and [15] for concepts about FPTAS. We then provide a straightforward dynamic program for the general case that runs in pseudopolynomial time in section 3. We use this dynamic program as an ingredient to design an algorithm that runs in polynomial time to solve a special case, where the inventory cost of the returned items is at least as much as the inventory cost of the manufactured items. In addition, we assume that the concave costs involved in manufacturing has a fixed cost and variable cost component. We also assume that the costs are time invariant.

In the single item economic lot-sizing problem with remanufacturing option (ELSR), we are given a time horizon T. Let $[T] := \{1, \ldots, T\}$. For each time period $t \in [T]$, we are given a demand D_t, and the amount of returned products R_t that is available for remanufacturing. W.l.o.g., we assume that manufacturing and remanufacturing can be both completed for an item in a single period. We also define the following cost functions for each time period $t \in [T]$:

1. manufacturing cost $f_t^m : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, for all $t = 1, \ldots, T$,
2. remanufacturing cost $f_t^r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, for all $t = 1, \ldots, T$
3. cost of holding manufactured items (we will refer to this a serviceable inventory items) $h_t^m : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, for all $t = 1, \ldots, T$ and
4. cost of holding returned items (we will refer to this a return inventory items) $h_t^r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, for all $t = 1, \ldots, T$.

We assume that costs are time dependent. Our main results in section 4 requires that that the costs are time invariant. Both inventory costs are linear. The concave cost structure associated with remanufacturing and manufacturing involves in a fixed cost and linear variable cost component , i.e., $f_t^i(x) = f_t^i + l_t^i x$, when $x > 0$ and 0 otherwise, for $i = \{r, m\}$. $f_t^r, f_t^m (l_t^r, l_t^m)$ are the fixed costs (linear variable costs) incurred in period t for remanufacturing and manufacturing respectively. We slightly abuse the notation here to denote both the fixed cost component and the function by the same notation, but this is easy to distinguish from the context. In each time period, we have the option to remanufacture the returned item, manufacture the item new, or use serviceable inventory from previous period to satisfy the demand. The problem requires a production plan that details the amount of products to be manufactured x_t, remanufactured y_t, the returned items carried in inventory p_t, and serviceable items carried in the inventory q_t, for each time period $t = 1, \ldots, T$ such that the demand is met in each time period and we minimize the total cost incurred. Excess returns from the production plan at the end of the planning period will just be disposed at no extra cost. We now give the formulation for this problem.

- $y_t(x_t)$: Amount of remanufactured (manufactured) item in time period t.
- $p_t(q_t)$: Amount of return (serviceable) inventory carried at time t
- $u_t^r(u_t^m)$: Binary variable indicating whether we remanufactured (manufactured) in period j

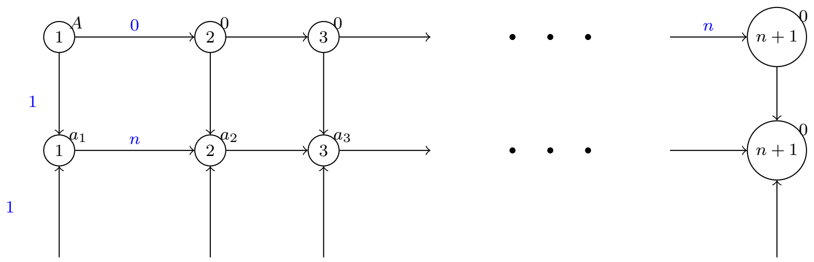


Fig. 1: Reduction from Partition problem

$$\min \sum_{t=1}^{T} (f_t^r u_t^r + l_t^r y_t + h_t^r p_t + f_t^m u_t^m + l_t^m x_t + h_t^m q_t) \tag{ELSR}$$
$$R_t + p_{t-1} = y_t + p_t, \quad \forall t \in [T]$$
$$q_{t-1} + y_t + x_t = D_t + q_t, \quad \forall t \in [T]$$
$$y_t \le M u_t^r, \quad \forall t \in [T]$$
$$x_t \le M u_t^m, \quad \forall t \in [T]$$
$$\mathbf{x}, \mathbf{y} \in \mathbb{R}^T_{\ge 0}$$
$$\mathbf{p}, \mathbf{q} \in \mathbb{R}^T_{\ge 0}$$
$$\mathbf{u} \in \{0,1\}^T$$

2 Complexity

The problem is known to be NP-hard in general [13,11]. We extend the reduction provided in [13] to show the following theorem. Note that the following proof had appeared in an unpublished work of the author [14] using a reduction from the partition problem. We repeat the proof here in this work. The problem could be viewed as non-time invariant or expected to have a zero outgoing return inventory at the end of the planning period.

Theorem 1 *ELSR does not have FPTAS unless P=NP.*

Proof. We will show this through a reduction from the partition problem, wherein we are given n integer $a_1, a_2, \ldots, a_n$. We want to determine if there exist a subset $S \subset \{1, ...n\}$ of integers such that $\sum_{i \in S} a_i = A$. In our reduction (see figure 1), we first take the time horizon $T = n+1$ and the demand for each time period $i = 1, \ldots, T-1 = n$ as a_i. The demand is zero for the last time period. We incur a fixed cost of 1 for both manufacturing and remanufacturing. The serviceable inventory cost is n for all time periods and returned inventory cost is 0 for the first n periods and it is n for the last period. The amount of returns in period 1 is $R_1 = A$ and there are no returns for all other time periods, $R_i = 0, i = 2, \ldots, n+1$. If there is a solution to the partition problem, i.e, YES instance, then the optimal solution to ELSR is less than n. We will use the returns A to satisfy the set of items adding up to A and the remaining items (which also add up to A) are satisfied by remanufacturing. Each period resulting in manufacturing or remanufacturing cost (but not both). If the partition problem is a no instance, i.e., then there are no subsets adding up to A, then we either need to manufacture and remanufacture at least in 1 period in addition to either manufacturing and remanufacturing in every other period or we would incur a serviceable inventory cost or the return inventory cost in the last period, resulting in a cost of at least $n+1$. This also rules out an FPTAS for the problem, since we can choose an $\epsilon < \frac{1}{n}$, say $\epsilon = \frac{1}{2n}$. Now, an algorithm that runs in $O(f(n, \frac{1}{\epsilon}))$, with $f(n, \frac{1}{\epsilon}) = f(n, 2n)$ being a polynomial function in n, provides an $(1+\epsilon)$-approximation for the ELSR that can distinguish YES and NO instances of the partition problem in polynomial time. We can transform a PARTITION instance into our ELSR instance. Then we can apply this approximation algorithm. If the output of this algorithm is at most n, then we can conclude the PARTITION instance as YES instance, since the approximation guarantee imply its solution has an objective value at most $n(1+\epsilon) < n(1+\frac{1}{n}) = n+1$ and we can only have integer solutions. Else we can conclude that the PARTITION instance is a NO instance. □

3 Dynamic program for the general case

We now provide a dynamic program that runs in pseudopolynomial time to solve the general case exactly. This is an extension of Wagner and Whitin's solution that incorporates the remanfacturing option, and we present it here as we will be needing it as an ingredient of our special case. We define the following function $W_t(p,q)$ as the minimum cost of obtaining a return inventory level of p and a serviceable inventory level of q at the end of period t, such that all demands are met for the periods $i = 0, \dots, t$ either through manufacturing new items or remanufacturing returns. We define the notation $\mathcal{D}_{i,j} := \sum_{t=i}^{j} D_t$ (corr. $\mathcal{R}_{i,j} := \sum_{t=i}^{j} R_t$) to denote the cumulative demands (corr. returns) between the periods i and j, for all $0 \le i \le j \le T$. We will define the inventory level sets $\mathcal{P}_t := \{0, \dots, \mathcal{R}_{1,t}\}$ and $\mathcal{Q}_t := \{0, \dots, \max\{\mathcal{R}_{1,t}, \mathcal{D}_{t,T}\}\}$. We will use the same notation but with a reduced state space for our special case. We will now do a forward recursion. We can now compute the value for $W_1(p,q)$, $p \in \mathcal{P}_1$, $q \in \mathcal{Q}_1$. For a specific value of p and q, there is exactly one way of obtaining the solution, so we can compute $W_1(p,q)$ for all possible values of p and q. In order to this, in our formulation ELSR, we need to calculate x_1^r and x_1^m by solving the set of equations $p + x_1^r = R_1$ and $D_1 + q = x_1^r + x_1^m$. Then $W_1(p,q) = f_1^r(x_1^r) + h_1^r(p) + f_1^m(x_1^m) + h_1^m(q)$. For infeasible solutions with $x_1^m < 0$ or $x_1^m < 0$, we set $W_1(p,q) = \infty$. Then, the recursive function is:

$$W_t(p,q) = \min_{\substack{\tilde{p} \in \mathcal{P}_{t-1} \\ \tilde{q} \in \mathcal{Q}_{t-1}}} [W_{t-1}(\tilde{p}, \tilde{q}) + f_t^r(\tilde{p} + R_t - p) + \tag{1}$$

$$f_t^m(D_t + q - \tilde{q} - (\tilde{p} + R_t - p)) + h_t^r(p) + h_t^m(q))] \tag{2}$$

The size of the state space of the problem is $T \cdot \mathcal{D}_{1,T} \cdot \mathcal{R}_{1,T}$, making the above algorithm pseudopolynomial in running time.

Theorem 2 *$W_t(p, q)$ is the optimal value of the ELSR problem for periods $1, \ldots, t$, when we need p (q) as the return (manufactured) inventory level at the end of time period t*

The proof is omitted as it is a straightforward extension from Wagner and Whitin [16] for the dynamic lot-sizing problem. As a consequence of Theorem 2, we get the following result.

Corollary 3. $\max_{p,q} W_T(p, q)$ *is the optimal solution to ELSR.*

4 Dynamic program for the special case: Return inventory cost is higher than serviceable inventory cost

We now investigate the special case where $h^r(p) \geq h^m(p)$, for all $p \in \mathbb{R}_{\geq 0}$. Generally, the serviceable inventory costs tend to be higher as the value of the products carried in the serviceable inventory is higher. In the special case, for instance, where the value of the returned products depreciate faster than a newly manufactured product or when there is no difference between a returned or a manufactured product, the problem has its applications. Note that we also omitted the time index as we are assuming the costs are time invariant. In the sequel, we show that, for this special cost structure, the sets $\mathcal{Q}_t$ and $\mathcal{P}_t$ are polynomially bounded. This in turn means that the the computation of $W_t(p, q)$ is polynomially bounded. For the sake of our analysis, we will introduce a special period, t^*, which is the last time period of remanufacturing in an optimal solution. We will present our analysis by separately bounding the state space, $\mathcal{Q}_t$ and $\mathcal{P}_t$, for the time periods before and after the time period t^*. It is used purely for the purposes of the proof. Let $(\mathbf{p}^*, \mathbf{q}^*)$ be the return and serviceable inventory levels in the optimal solution where we assumed t^* and ℓ^* to be known in the analysis. We define the following notation for compactness: $(a - b)^+ := \max\{a - b, 0\}$

4.1 Bounding state space of $\mathcal{P}$ and $\mathcal{Q}$

Lemma 1 *For the optimal solution $\mathbf{p}^*, \mathbf{q}^*$, let $\hat{R}_t := p^*_{t-1} + R_t$ be the returned goods available at some time period $t < t^*$, then there is an optimal solution (by possibly re-writing the solution $\mathbf{p}^*, \mathbf{q}^*$ from time t and onwards) in which the amount of remanufactured items in time t will only be from the set $\{0, \hat{R}_t\}$. If such a choice of return inventory is not possible, then we can create a new optimal solution with t being the last remanufactured period.*

Proof. Suppose in $(\mathbf{p}^*, \mathbf{q}^*)$, we produce something not from this set $\{0, \hat{R}_t\}$. Hence, some intermediate return stock of $0 < a < \hat{R}_t$ is carried, which also means that we are remanufacturing at time t in the optimal solution. Since $t < t^*$, there exists a time period after t in the optimal solution where we remanufacture. Let the $\tilde{t}$ be the first time period after t, when we remanufacture in the optimal solution. We are also carrying a non-zero return inventory until this time period. If we remanufacture at least a in time $\tilde{t}$, then we could have

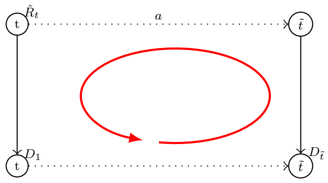


Fig. 2: A rerouting the returns remanufactured at $\tilde{t}$ to time period t

remanufactured this a in time t and carried a units of manufactured inventory until time $\tilde{t}$ with no additional cost, since return inventory cost is higher than manufactured inventory cost. If we produced less than a in time $\tilde{t}$, say $\tilde{a}$, then we could have produced $\tilde{a}$ in time t and produced nothing in time $\tilde{t}$ and continue with our argument (with $a - \tilde{a}$ being the new a and the next time period of remanufacturing being the new $\tilde{t}$). If $\tilde{t} = t^*$ and $\tilde{a} < a$, then we would have new optimal solution with t being the last time period of remanufacturing. □

An alternative way of interpreting the above lemma is that whenever we choose to remanufacture at time t, we remanufacture all return inventory available or nothing. We can reroute the returns that were remanufactured at time $\tilde{t}$ to time t (see figure 2) and get a new production plan with better or same cost. The above lemma gives us the following lemma and corollary.

Corollary 4. *For each time period $t < t^*$, there exists an optimal solution for ELSR where the possible inventory level of the returned products right after period t takes a value only from the set $\{0, \mathcal{R}_{i,t}\}$, for all $i = 1, \ldots, t$, where t^* is the last time period of remanufacturing in that optimal solution.*

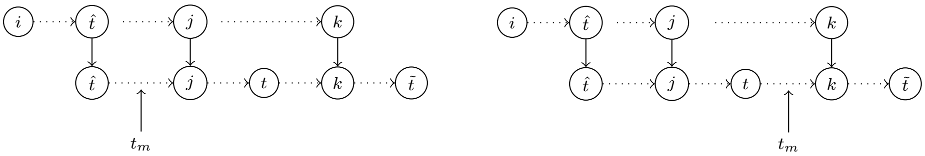
Proof. The proof can be obtained through induction by invoking Lemma 1 and the induction hypothesis. □

Corollary 5. *For each $t < t^*$, the total number of return inventory levels to be kept is less than t^2.*

We need the following intermediate lemmas before we give the main proof.

Lemma 2 *There exists an optimal solution in which between two successive manufacturing periods (say t and $t + j$), the serviceable inventory falls to zero, i.e., the outgoing serviceable inventory is zero for at least one period between t and $t + j - 1$.*

Proof. Let x be the smallest inventory level between the periods t and $t + j$ and y be the amount of manufactured items in time t. We now reduce the inventory level of all time periods between t and $t + j$ by $\min(x, y)$ and decrease the manufactured items in time t by $\min(x, y)$ and increase the manufactured item in time $t + j$ by $\min(x, y)$. If we reduced the manufactured items in t to zero, then we repeat the argument with $t + j$ and the period before t when we manufactured. Note that this procedure results in a new solution with a



(a) Case: $\hat{t} \leq t_m \leq t$ (manufacturing happens before time t)

(b) Case: $t \leq t_m \leq \tilde{t}$ (either manufacturing happens after t or does not happen)

Fig. 3: Bounding serviceable inventory levels for period $t < t^*$

better cost and with at least one time period with zero serviceable inventory between two successive manufacturing periods. □

Definition 1 We call a continuous interval of periods $\hat{t}, \ldots, \tilde{t}$, as non-zero serviceable inventory interval (NSI), if the following conditions are true

1. Incoming serviceable inventory of $\hat{t}$ is zero
2. Outgoing serviceable inventory of $\tilde{t}$ is zero
3. Finally, we must have non-zero serviceable inventory at all periods $\hat{t}, \hat{t}+1, \ldots, \tilde{t}-1$. By this, we mean that the serviceable inventory never drops to zero within a NSI.

Lemma 3 *There exists an optimal solution, where the serviceable inventory level at time period t in a given NSI, $[\hat{t}, \tilde{t}]$, will be in the set*

$$\{0, \mathcal{D}_{t+1,\tilde{t}} + \mathcal{R}_{j+1,k}, \mathcal{R}_{i,j} - \mathcal{D}_{\hat{t},t}\}$$

where $j \in [\hat{t}, t]$, $k \in [t, \tilde{t}]$ and $i \in [1, \hat{t}]$.

Proof. Let us consider the NSI $[\hat{t}, \tilde{t}]$, in which t is present. From lemma 2, if manufacturing took place at some time t_m, then it will be only period of manufacturing in its NSI (see figure 3). We define the following time periods

1. let $i-1$ be last period before $\hat{t}$ when remanufacturing takes place
2. let $j \geq \hat{t}$ be the last period before t when remanufacturing takes place
3. let $t \leq k \leq \tilde{t}$ be the last period before $\tilde{t}$, when remanufacturing take place

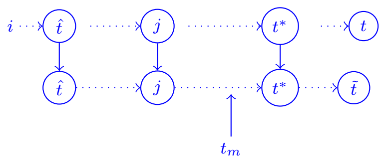
Case $\hat{t} \leq t_m \leq t$: The serviceable inventory level at time t is $x + \mathcal{R}_{i,j} - \mathcal{D}_{\hat{t},t}$, where x is the amount manufactured in time t_m. This is true because all return items between period i and j gets remanufactured before t (and those are the only items that get remanufactured from the definition of i and j - see figure 3a) and all demands between the periods $\mathcal{D}_{\hat{t},t}$ gets deducted from the total amount manufactured and remanufactured. Now x takes the value $\mathcal{D}_{\hat{t},\tilde{t}} - \mathcal{R}_{i,k}$ as all returns between the periods i and k gets remanufactured and all remaining demand between the periods in NSI has to then come from the only manufacturing period in the NSI.

Case $t < t_m \leq \tilde{t}$ or when no manufacturing happens in the NSI: The serviceable inventory level at t will be $\mathcal{R}_{i,j} - \mathcal{D}_{\hat{t},t}$. This follows similar reasoning to the previous case but this time without x (see figure 3b).

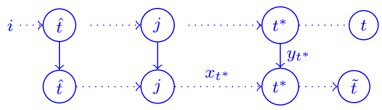
□

Lemma 4 *For each t, the number of serviceable inventory levels kept is at most $t^2(T-t) + t^3$.*

Proof. We will first work out the number of distinct possible values that $\mathcal{D}_{t+1,\tilde{t}} + \mathcal{R}_{j+1,k}$ can take. $\hat{t}$ is not explicitly appearing in the expression. It indirectly determines the possible values that j can take. $\hat{t}$ values can be between 1 and t. This would imply for a given t, $j = 1$ to t, $k = t$ to $\tilde{t}$ and $\tilde{t} = t$ to T. When $\tilde{t} = T - r$, there are t possible values for j and $T - r - t$ possible values for k. This needs to be summed up for $r = 1$ to t possible values of $\tilde{t}$. Thus the number of distinct possible values that $\mathcal{D}_{t+1,\tilde{t}} + \mathcal{R}_{j+1,k}$ can take is at most $\sum_{r=1}^{t} t(T-r-t) \leq t^2(T-t)$.
By a similar reasoning, with $i = 1$ to $\hat{t}$ and $\hat{t} = 1$ to t, the number of distinct values that $\mathcal{R}_{i,j} - \mathcal{D}_{\hat{t},t}$ can take are $\sum_{r=1}^{t}(t^2 - r)r \leq t^3$. □



(a) Manufacturing happens in NSI. Return inventory at $t^* = 0$



(b) No manufacturing in the NSI ($x_{t^*} = \mathcal{R}_{i,j} - \mathcal{D}_{\hat{t},t^*-1}, y_{t^*} = \mathcal{D}_{\hat{t},\tilde{t}} - \mathcal{R}_{i,j}$)

Fig. 4: Return inventory at time $t \geq t^*$

Lemma 5 *Given the NSI $[\hat{t}, \tilde{t}]$ that contains t^*, the return inventory level in the periods $t \geq t^*$ will be in the set*

$$\{0, \mathcal{R}_{t^*+1,t}, \mathcal{R}_{i,t} - \mathcal{D}_{\hat{t},\tilde{t}}\}$$

where $i \in [1, \hat{t}]$.

Proof. We will divide the analysis into 2 cases:
Manufacturing takes place in t^*'s NSI: In this case the return inventory level at t^* is then 0. This is because, we have an uncapacitated concave cost network flow problem whose optimal solution is an extreme flow. Every flow can be decomposed into paths and cycles and extreme flows do not contain a cycle (see [2]). If manufacturing happens and we have return inventory level at t^* non-zero, we will end up inducing a cycle in the flow based solution. As this

cannot happen, we can rule out the possibility of returns at t^* being non-zero for this case. This would imply that the return inventory level at t will be $\mathcal{R}_{t^*+1,t}$.

No manufacturing happens in t^*'s NSI: Let

- $\hat{t}$ and $\tilde{t}$ be the starting and ending time period of the NSI in which t^* is present
- $i-1$ be the last period before $\hat{t}$, when remanufacturing happened.
- j be the last period before t^*, when remanufacturing happened.

The incoming serviceable inventory level at t^* is then $\mathcal{R}_{i,j} - \mathcal{D}_{\hat{t},t^*-1}$ obtained through remanufacturing the returns between periods i and j that are used to satisfy demands between $\hat{t}$ and t^*-1 (see figure 4b). This incoming serviceable inventory level along with the remanufactured returns at time period t^* (say y_{t^*}) must satisfy the demands between t^* and $\tilde{t}$. In other words $y_{t^*} + \mathcal{R}_{i,j} - \mathcal{D}_{\hat{t},t^*-1} = \mathcal{D}_{t^*,\tilde{t}}$. In this case, the return inventory level at t will be

$$\begin{aligned}\mathcal{R}_{t^*+1,t} + (\mathcal{R}_{j+1,t^*} - y_{t^*}) &= \mathcal{R}_{t^*+1,t} + (\mathcal{R}_{j+1,t^*} - (\mathcal{D}_{t^*,\tilde{t}} - (\mathcal{R}_{i,j} - \mathcal{D}_{\hat{t},t^*-1})))\\ &= \mathcal{R}_{i,t} - \mathcal{D}_{\hat{t},\tilde{t}}\end{aligned}$$

□

Corollary 6. *For $t \geq t^*$, the number of return inventory levels is bounded by $\mathcal{O}(T^3)$.*

From corollary 5 and lemma 4, we have the following lemma.

Lemma 6 *For $t \leq t^*$, the $\mathcal{P}_t \times \mathcal{Q}_t$ is bounded by $\mathcal{O}(T^5)$.*

From corollaries 5 and 6, we have the following lemma.

Lemma 7 *For $t \geq t^*$, the $\mathcal{P}_t \times \mathcal{Q}_t$ is bounded by $\mathcal{O}(T^5)$.*

From (2), we have that each state space evaluation taking $\mathcal{O}(T^5)$. This can be substantially reduced as we see in the subsequent lemmas.

Lemma 8 *At time period t and for each $p \in \mathcal{P}_t$ and $q \in \mathcal{Q}_t$, we need to consider at most $\mathcal{O}(T)$ $(\tilde{p}, \tilde{q}) \in \mathcal{P}_{t-1} \times \mathcal{Q}_{t-1}$ to consider in the evaluation of $W_t(p,q)$ in (2).*

Proof. When $t-1$ and t are in a different NSI, the only $\tilde{q}$ we need to consider is 0 from the definition of a NSI. If remanufacturing happens at t (i.e. $p=0$ from lemma 1), then there are $\mathcal{O}(T)$ possibilities for $\tilde{p}$ depending on when remanufacturing happened before $t-1$. If remanufacturing did not happen at t, we would then consider a fixed return value for p, say $\mathcal{R}_{i,t}$, corresponding to a period i, when remanufacturing happened before t. In this case, there is unique value for $\tilde{p}$, which is either 0 when $i = t-1$ or $\mathcal{R}_{i,t-1}$ when $i < t-1$. We will now discuss the case when $t-1$ is in the same NSI as t. For a given NSI $[\hat{t}, \tilde{t}]$, at t we have $q \in \{0, D_{t+1,\tilde{t}} + \mathcal{R}_{j+1,k}, \mathcal{R}_{i,j} - \mathcal{D}_{\hat{t},t}\}$, for all $j \in [\hat{t}, t], k \in [t, \tilde{t}]$

and $i \in [1, \hat{t}]$. At $t-1$ we have $\tilde{q} \in \{0, D_{t,\tilde{t}} + \mathcal{R}_{\bar{j}+1,\bar{k}}, \mathcal{R}_{\bar{i},\bar{j}} - \mathcal{D}_{\hat{t},t}\}$ for all $\bar{j} \in [\hat{t}, t-1], \bar{k} \in [t-1, \tilde{t}]$ and $\bar{i} \in [1, \hat{t}]$. However, when we consider a specific value for q, we observe the following definitions for the periods i and k (See (1) and (3) in proof of lemma 3):

1. $t \leq k \leq \tilde{t}$ is the last period before $\tilde{t}$ when remanufacturing takes place
2. $i-1$ is the last period before $\hat{t}$ when remaufacturing takes place

From the first (resp. second) observation, we have that $\bar{k}$ (resp. $\bar{i}$) coincides with k (resp. i) as these definitions have to be true with respect to $t-1$ as well for a given NSI. So, we only need to consider $\mathcal{O}(T)$ values for $\tilde{q}$ corresponding to $\bar{j} \in [\hat{t}, t-1]$, when we consider a given value of q. By definition, j is the last period before t, when remanufacturing happened. So p takes a value in the set $\{0, \mathcal{R}_{j,t}\}$. The case $p = 0$ corresponds to remanufacturing happening in p. In this case, there is a unique value for $\tilde{p}$, which is $\mathcal{R}_{\bar{j},t-1}$ corresponding to the same value of $\bar{j} \in [\hat{t}, t-1]$ that we used for $\tilde{q}$. When we consider $p = \mathcal{R}_{j,t}$ we still have a unique value for $\tilde{p}$: either $\tilde{p} = 0$ when $j = t-1$ or $\tilde{p} = \mathcal{R}_{j,t-1}$ when $j < t-1$. $\square$

Lemma 6, 7 and 8 gives us the following Theorem.

Theorem 7 *The overall complexity of the algorithm is $\mathcal{O}(T^7)$.*

5 Conclusion and open problems

In this work, we studied the ELSR problem. We first provided a hardness proof that rules out FPTAS for this problem in the general case. We then provided a dynamic program with a pseudopolynomial running time to solve the general version of the problem. We later showed how this can be used to design a polynomial running time algorithm, when we make some assumptions on the cost structure. A number of open problems still remain to be solved. Although we have ruled out a possibility of FPTAS for the general case, we have no proofs for APX-hardness (see [15]) or lack of FPTAS for the time invariant case. The polynomial time algorithm presented only works for fixed cost structure. For general concave costs, we do not yet know the complexity. The algorithm itself is not practical for large instances but knowing that it is tractable would give incentives to look for linear programming representations for these problems.

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