Is Stochastic Volatility relevant for Dynamic Portfolio Choice under Ambiguity?[*]

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Abstract

Literature on dynamic portfolio choice has been finding that volatility risk has low impact on portfolio choice. For example, using long-run U.S. data, Chacko and Viceira (2005) found that intertemporal hedging demand (required by investors for protection against adverse changes in volatility) is empirically small even for highly risk-averse investors. We want to assess if this continues to be true in the presence of ambiguity. Adopting robust control and perturbation theory techniques, we study the problem of a long-horizon investor with recursive preferences that faces ambiguity about the stochastic processes that generate the investment opportunity set. We find that ambiguity impacts portfolio choice, with the relevant channel being the return process. Ambiguity about the volatility process is only relevant if, through a specific correlation structure, it also induces ambiguity about the return process. Using the same long-run U.S. data, we find that ambiguity about the return process may be empirically relevant, much more than ambiguity about the volatility process. Anyway, intertemporal hedging demand is still very low: investors are essentially focused in the short-term risk-return characteristics of the risky asset.

Keywords: Dynamic Portfolio Choice, Stochastic Volatility, Ambiguity, Robust Control, Perturbation Theory.

JEL Classification: C61 · D81 · E21 · G11.

1 Introduction

We study optimal dynamic portfolio choice under a stochastic investment opportunity set, of an investor that is averse both to risk and ambiguity. We want to understand if and how ambiguity about

the stochastic processes that generate the return and volatility of the risky asset impacts portfolio choice. More particularly, we want to assess if stochastic volatility continues to have a low impact on portfolio choice, as it has been found in the literature, in the presence of ambiguity about the stochastic investment opportunity set.[1]

There is a large literature on portfolio choice (see, e.g., Kogan and Uppal (2001) and Campbell and Viceira (2002) for a survey), but relatively few works study optimal dynamic portfolio choice with stochastic variance of the risky asset's return. Some examples are Kim and Omberg (1996) and Chacko and Viceira (2005), with incomplete markets, and Schroder and Skiadas (1999), with complete markets. Schroder and Skiadas (2003) gave a general closed-form solution for the consumption-portfolio problem, which includes the other models as special cases. Other papers consider multiple risky assets, as Liu (2007) and Buraschi et al. (2010). Potentially adverse changes in the investment opportunity set are associated with stochastic variance of the risky asset's return, which therefore represents a source of risk to investors. This implies, from Merton (1973), that stochastic variance originates an intertemporal hedging demand.[2] Chacko and Viceira (2005) concluded, using long-run U.S. data, that this intertemporal hedging demand is empirically small even for highly risk-averse investors.

In all the papers mentioned above, there is only risk, and no ambiguity. Ambiguity is uncertainty that cannot be represented by a single probability distribution. Risk, on the contrary, is uncertainty that is susceptible of being described by a probability distribution. This conceptual distinction, first explored by Knight (1921), has relevant implications for the behavior of economic agents, and, therefore, for economic theory in general. Ellsberg (1961) disclosed experimental evidence supporting the Knightian distinction between risk and ambiguity. This evidence became known as the Ellsberg paradox, and motivated a huge literature (surveyed in Camerer and Weber (1992) and Epstein and Schneider (2010)).

Notwithstanding this, the mainstream theory of choice under uncertainty in economics ignored ambiguity for several decades, remaining based on the expected utility theory of von Neumann and Morgenstern (1944), where the probabilities of the possible states of nature are known, and on the subjective expected utility theory of Savage (1954), where, although probabilities are not necessarily known, the choice behavior of an agent coincides with the maximization of expected utility according to some subjective probability beliefs.

Gradually, ambiguity is being incorporated in decision theory. Two main approaches are being used: (i) the multiple priors (MP) approach, where the single probability measure of the expected utility models (precise beliefs) is replaced by a set of probabilities or priors (imprecise beliefs); (ii) the robust control (RC) approach, associated to an assumption of model uncertainty. The relationship between the MP and RC approaches has been widely discussed in the literature, for example, in Hansen and Sargent (2001), Hansen et al. (2002), Epstein and Schneider (2003), and Maccheroni et al. (2006).

Ahn et al. (2011) found empirical support for the relevance of studying the portfolio choice problem under ambiguity (about 2/3 of agents in their experience showed a positive degree of ambiguity aversion). Bossaerts et al. (2010) also concluded that ambiguity aversion can be observed in competitive

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[1] Throughout this paper, by "volatility" of the risky asset we mean the variance of the risky asset's return. For mathematical convenience, we work with precision (the reciprocal of variance).

[2] In the multivariate setting of Buraschi et al. (2010), with a stochastic variance-covariance matrix, there is an intertemporal hedging demand associated with the stochastic variance and another associated with the stochastic correlation between the returns of the risky assets.

markets and that it influences portfolio choice and asset prices.

In studies of portfolio choice with ambiguity, Garlappi et al. (2007) and Gollier (2011) concluded that, by introducing ambiguity aversion in a static MP approach, the optimal demand for the risky asset decreases versus the standard mean-variance and Bayesian models.[3] The same conclusion was reached in a dynamic MP setting (e.g., Chen et al. (2011)) and in a dynamic RC model (e.g., Maenhout (2004, 2006) and Xu et al. (2011)). The implications of ambiguity aversion for portfolio diversification have also been studied (Uppal and Wang (2003)). In all these works, with the exception of Xu et al. (2011), the source of ambiguity is exclusively the expected risky asset's return or the risky asset's return process.

In this paper, we extend the model of Chacko and Viceira (2005) for optimal dynamic portfolio choice, by introducing ambiguity about the data generating process of the stochastic investment opportunity set. The motivation for this is provided by Chacko and Viceira themselves:

"*An important caveat of our empirical analysis is that we have counterfactually assumed that investors observe volatility (or precision), and that they take as true parameters our empirical estimates of the joint process for returns and volatility. In practice, however, investors do not observe volatility, and they do not know the parameters of the process for volatility, or even the process itself.*"

Literature on dynamic portfolio choice with stochastic variance has been finding that variance risk has low impact on portfolio decisions (e.g., Chacko and Viceira (2005) and Liu (2007)). We want to understand if this continues to be true if uncertainty is considered in a broader perspective, by taking into account an "ambiguity dimension" alongside the standard "risk dimension".

It has been advocated in the literature (Cao et al. (2005), Garlappi et al. (2007) and Ui (2011)) that it is reasonable to assume that investors estimate the variance of the risky asset's return without ambiguity, and that it is preferable to assume ambiguity about expected returns. Reasons invoked for this are analytical tractability, empirical evidence on the predictability of the variance of stock returns (Bollerslev et al. (1992)), higher difficulty in estimating the expected returns versus expected variance (Merton (1980)) and higher costs associated with errors in estimating expected returns versus expected variance (Chopra and Ziemba (1993)).

Nevertheless, we introduce ambiguity also about the variance process of the risky asset's return because (i) there is no *a priori* reason to assume that investors are not ambiguous about it, and because (ii) we are able to find an asymptotic analytical solution and test it empirically.

In Faria et al. (2009), the setting of Chacko and Viceira (2005) was extended by considering a representative investor that is ambiguous about one specific parameter of the stochastic variance process (the expected value). A MP approach was adopted, and the conclusion was that ambiguity does not impact the instantaneous optimal portfolio choice rule. The ambiguity effect would only exist if the investor were not able to continuously update his portfolio. In Faria and Correia-da Silva (2010), we obtained the optimal portfolio rule in a dynamic setting, with stochastic variance and ambiguity about its process. There, it was assumed that the representative investor derives utility exclusively through terminal wealth, implying that the intertemporal consumption-savings decision is ignored, and ambiguity is treated through a RC approach. The optimal portfolio rule that was derived showed that ambiguity aversion has an additive impact to risk aversion.

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[3]Although the result of Gollier (2011) requires some restrictions on the set of priors and on the investor's attitude towards risk.

The closest paper to the present work is that of Xu et al. (2011), where preferences of the representative investor are given by the SDU function introduced by Duffie and Epstein (1992b) and ambiguity about the data generating process with stochastic variance is also considered and studied through a RC approach. Compared with the contribution of Xu et al. (2011), our paper brings three major novelties. The first results from the fact that we adopt a different RC methodology: "constrained preferences" instead of "multiplier preferences" (also applied in Faria and Correia-da Silva (2010)).[4] Under "constraint preferences", there is a constraint on the magnitude of the allowable perturbations from the benchmark model. Under "multiplier preferences", preferences for robustness are constructed by penalizing deviations from the benchmark model, with higher deviations being more penalized than smaller ones. A relevant implication is that under "constrained preferences" the impact of ambiguity on the optimal portfolio choice is more than simply an enhanced risk aversion. Moreover, in order to derive optimal policies under ambiguity, we use perturbation theory, as, for example, in Trojani and Vanini (2002, 2004). The rationale behind the perturbation (asymptotic) method is well described by Trojani and Vanini (2004): "[...] *formulate a general problem, find a particular relevant case that has a known solution, and use this as a starting point for computing the solution to nearby problems.*" In our case, as in Trojani and Vanini (2004), the asymptotic solution of the problem under ambiguity holds in neighborhoods of a model with no ambiguity aversion.

The second difference relatively to Xu et al. (2011) is that we want to understand the relevant channels (return process, variance process or both) through which ambiguity impacts dynamic portfolio choice. For that, we study optimal dynamic portfolio choice when ambiguity is simultaneously about the return and volatility processes, as in Xu et al. (2011), and when it is exclusively about the return process or the variance process.

The third difference versus Xu et al. (2011) is that we simulate our model using long-run U.S. data to measure the empirical significance of the impact of ambiguity on optimal portfolio choice. This is crucial, as, ultimately, we are addressing the question of whether stochastic variance is relevant for portfolio choice.

The main conclusions of this paper concern the impact of ambiguity on optimal dynamic policies, both when ambiguity is simultaneously about the return and variance processes and when it is exclusively about one of these stochastic processes. In all scenarios, we find that ambiguity does not impact the optimal consumption rule (instantaneous consumption as a function of current wealth). The effect of ambiguity is a reduction of the demand for the risky asset. The relevant channel is the return process, as when ambiguity is exclusively about the variance process there is no impact on the optimal portfolio rule. Ambiguity about the variance process is only relevant if, through a specific correlation structure, it also induces ambiguity about the stochastic process that generates the return of the risky asset.

Using long-run U.S. data, we find that ambiguity about the stochastic processes driving the investment opportunity set is empirically relevant for portfolio decisions. Our simulation suggests that ambiguity about the return process is empirically much more relevant than ambiguity about the variance process. We also conclude that, even accounting for ambiguity about the variance process, the intertemporal hedging demand (required by investors for protection against adverse changes in vari-

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[4]We adopt this terminology from Hansen and Sargent (2006).

ance) is still very low. Investors are essentially focused in short term risk-return characteristics of the risky asset. This had been found under settings were uncertainty is exclusively risk (for example in Chacko and Viceira (2005) and Liu (2007)) and we extend that conclusion for a setting where uncertainty also has an ambiguity dimension.

The paper is organized as follows. In section 2, we present the model and state the problem to be solved. In section 3, we present the analytical solution to that problem and the key results. In section 4 we analyze alternative scenarios for the sources of ambiguity, deriving analytical solutions and comparing its results with those of section 3. In section 5, simulation results are presented. In section 6, we conclude the paper with some remarks.

2 Consumption and Portfolio Choice Problem

In section 2.1, the investment opportunity set is described. In section 2.2, the preferences of the representative investor are presented. In section 2.3, the dynamic optimization problem to be solved is disclosed.

2.1 Investment Opportunity Set

In this section, we describe the investment opportunity set that is faced by the representative investor. We follow closely Chacko and Viceira (2005).

All wealth must be allocated between a riskless asset with price B_t and a risky asset with price S_t. The instantaneous return of the riskless asset is described by:

$$\frac{dB_t}{B_t} = rdt \,, \tag{1}$$

where r stands for the risk free interest rate.

The instantaneous return of the risky asset is given by:

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{\frac{1}{y_t}} \left(\rho dW_y + \sqrt{1 - \rho^2} dW_\varepsilon \right) \,, \tag{2}$$

where μ is the expected return of the risky asset and y_t is the instantaneous precision of the risky asset's return process (the instantaneous variance is $v_t = \frac{1}{y_t}$). W_ε and W_y are two independent standard Brownian motions.

The precision, y_t, follows a mean-reverting, square-root process as used by Cox et al. (1985):

$$dy_t = \kappa \left(\theta - y_t \right) dt + \sigma \sqrt{y_t} dW_y \,, \tag{3}$$

where the expected value of precision is $E\left[y_t\right] = \theta$, the reversion parameter is $\kappa > 0$, and, thus, $Var\left[y_t\right] = \frac{\sigma^2 \theta}{2\kappa}$. To guarantee standard integrability conditions, it is assumed that $2\kappa\theta > \sigma^2$, as in Cox et al. (1985).

Applying Itô's Lemma to (3), a mean-reverting, square-root process for proportional changes in

variance is obtained:

$$\frac{dv_t}{v_t} = \kappa_v \left(\theta_v - v_t \right) dt - \sigma \sqrt{v_t} dW_y \,, \tag{4}$$

where $\theta_v = \left(\theta - \frac{\sigma^2}{\kappa} \right)^{-1}$ and $\kappa_v = \kappa \left(\theta - \frac{\sigma^2}{\kappa} \right) = \frac{\kappa}{\theta_v}$.

An approximation of the unconditional mean of instantaneous variance is:[5]

$$E\left[v_t\right] \approx \frac{1}{\theta} + \frac{\sigma^2}{2\kappa\theta^2} \,. \tag{5}$$

As the expected return of the risky asset, μ, is assumed to be constant, (5) is also the unconditional variance of the risky asset's return. Chacko and Viceira (2005) performed a Monte Carlo simulation to validate this statement and the accuracy of the approximation, having concluded that (5) understates the true variance by 0.27%.

It is implicit in (2)-(3) that shocks in precision (W_y) are correlated with shocks in the return of the risky asset, with correlation given by $\rho > 0$. From (4), this implies that the instantaneous correlation between proportional changes in the risky asset's return and variance is given by:

$$Corr_t \left(\frac{dv_t}{v_t}, \frac{dS_t}{S_t} \right) = -Corr_t \left(dy_t, \frac{dS_t}{S_t} \right) = -\rho dt \,. \tag{6}$$

This investment opportunity set incorporates three of the main stylized facts about the variance of the return of risky assets: the mean reversion property, the "leverage effect" property (given by the negative correlation between return and its variance), and the fact that proportional changes in variance are higher when variance is high.

2.2 Investor Preferences

It is assumed that the representative investor is not totally sure about the stochastic processes (2)-(3) that generate the dynamic investment opportunity set. In other words, the uncertainty faced by the representative investor has two dimensions: risk and ambiguity.

Additionally, it is assumed that the preferences of the representative investor are described by the stochastic differential utility (SDU) function introduced by Duffie and Epstein (1992b) and applied to asset pricing theory in Duffie and Epstein (1992a). This is a continuous-time form of recursive utility, analogous to the discrete-time parametrization of Epstein and Zin (1989, 1991), that exhibits intertemporal consistency, admits Bellman's characterization of optimality, and separates risk aversion from elasticity of intertemporal substitution.

The utility process that defines the SDU function is represented by:

$$J = E_t \left[\int_t^\infty f\left(C_s, J_s\right) ds \right] \,, \tag{7}$$

where C_s represents current consumption and J_s is the continuation utility for the consumption flow C, at time $t = s$, with infinite time horizon. In our setting, the function $f\left(C_s, J_s\right)$ is the normalized

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[5]Obtained by taking expectations of the second-order Taylor expansion of v_t around θ (Chacko and Viceira (2005)).

aggregator that generates J, defining a SDU function that represents the preferences introduced by Kreps and Porteus (1978). An explicit closed-form expression for that SDU utility function is not available.

We assume a unitary elasticity of intertemporal substitution ($\psi = 1$), because: (i) with $\psi = 1$ there is an exact solution of the Bellman equation that we will obtain; and (ii) we conclude that the main analytical and empirical results do not change if $\psi \neq 1$.[6] With $\psi = 1$, the normalized aggregator $f(C, J)$ takes the form (e.g, Duffie and Epstein (1992a,b)):

$$f(C, J) = \beta (1 - \gamma) J \left\{ ln (C) - \frac{1}{1 - \gamma} ln [(1 - \gamma) J] \right\}, \tag{8}$$

where $\gamma > 0$ is the coefficient of relative risk aversion and $\beta > 0$ is the rate of time preference. If $\gamma = 1$, (8) can be replaced by the standard log-utility representation.

A remark regarding the preference for the timing of the resolution of risk. With the preference structure of Kreps and Porteus (1978), investors can have preference for early or late resolution of risk (as well as indifference), while the standard additive intertemporal utility function implies that investors are indifferent to the temporal resolution of risk. In the framework of Epstein and Zin (1989), the preference for temporal resolution of risk depends on the relationship between ψ and γ: if $\gamma > \frac{1}{\psi}$ ($<, =$) investors have preference for early (late, indifferent) resolution of risk. Our specification (7) from Duffie and Epstein (1992a), being the continuous-time limit of Epstein and Zin (1989), inherits this property. However, on the contrary of other streams of literature with Epstein-Zin preferences, for example, the "long-run risk" literature (from the seminal work of Bansal and Yaron (2004)), we do not restrict the investor to have preference for early resolution of risk. Two main reasons support this decision: (i) as our model evolves in a long-run setting, the possibility of the "cost" becoming higher than the "benefit" of planning advantages brought by the early resolution of risk (Arai (1997)) should not be excluded and (ii) there is evidence that investors may have preference for late resolution of risk (Epstein and Zin (1991)).

2.3 Dynamic Optimization Problem

Ambiguity about the investment opportunity set is studied with robust control (RC) techniques, firstly introduced in economics by Hansen and Sargent (1995). The representative investor has a reference model, but, facing ambiguity about the true model, considers a family of alternative models that are statistically difficult to distinguish from his benchmark.

Under the RC approach, two main formulations have been used in the ambiguity related literature: the "constraint preferences" and the "multiplier preferences". Under "constraint preferences" (e.g., in Hansen et al. (2006)), there is a constraint on the magnitude of the allowable perturbations from the benchmark model. Under "multiplier preferences" (e.g., in Maenhout (2006)), preferences for robustness are constructed by penalizing deviations from the benchmark model, with higher deviations being more penalized than smaller ones. Although both settings are related, through the Lagrange Multiplier Theorem (Hansen and Sargent (2006)), they end up being structurally very different.

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[6] These results for $\psi \neq 1$ are not presented, for economy of space, but they are available on request. Chacko and Viceira (2005) present an approximate solution for the Bellman equation that is obtained if $\psi \neq 1$. That solution converges to the exact solution when $\psi = 1$.

One difference is that under the "constraint preferences" RC approach, the specification of the ambiguity aversion can be based on a rectangular set of priors, which guarantees a dynamically consistent preference ordering. In those cases, preferences can be represented by the recursive multiple priors utility (RMPU) specification (Chen and Epstein (2002) and Epstein and Schneider (2003)).[7]

Additionally, the "constraint preferences" approach enables ambiguity to expand the range of qualitative behavior that can be rationalized versus the standard expected utility theory (as pointed out by Epstein and Schneider (2010)). This contrasts with the "multiplier preferences" approach, which is observationally equivalent to expected utility theory: it enables reinterpretations of some results obtained under the expected utility theory, that can be quantitatively more appealing,[8] but does not enlarge the spectrum of qualitative behavior that can be rationalized. In the "multiplier preferences" RC approach, ambiguity aversion is in practice translated into an enhanced level of risk aversion (as concluded for optimal dynamic portfolio choice in Maenhout (2004, 2006), Faria and Correia-da Silva (2010) and Xu et al. (2011)).

In this paper, we adopt a "constraint preferences" RC approach as in Faria and Correia-da Silva (2012). The investor considers contaminations (alternative models), P^h, around his reference belief, P, under which processes (2)-(3) evolve. The contaminations are assumed to be absolutely continuous with respect to P, and, therefore, are equivalently described by contaminating drift processes, $h = \begin{bmatrix} h^y & h^\varepsilon \end{bmatrix}^\top$, that contaminate the vector of Brownian motions, $W = \begin{bmatrix} W_y & W_\varepsilon \end{bmatrix}^\top$, associated with the stochastic processes that generate the risky asset's return and volatility. In an alternative model, P^h, the Brownian motion is $W^h(t) = W(t) + \int_0^t h(s)\,ds$.[9]

An upper bound is imposed on the contaminating drift processes:

$$h^\top h \leqslant 2\eta, \tag{9}$$

where $\eta \geqslant 0$ is a parameter that can be interpreted as the level of ambiguity. The class of admissible Markovian drift contaminations satisfying this entropy bound (9) is denoted by $\mathcal{H}$.

Alternative models should be statistically close to the reference model. Otherwise, the agent would be able to distinguish them and, consequently, would not face ambiguity. This means that η must be small. Moreover, the bound (9) constrains both the instantaneous time variation and the continuation value of the relative entropy between the reference belief, P, and any admissible contaminated belief, P^h. Trojani and Vanini (2004) explain that the set $\{h : h^\top h \in [0, 2\eta], \forall t \geqslant 0\}$ defines a rectangular set of priors because any process h (and therefore any probability measure P^h) in this set corresponds to a selection of transition densities from t to $t + dt$, $t \geqslant 0$, such that $h^\top h \in [0, 2\eta]$.[10]

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[7]Rectangularity is the property that allows updating every prior under the recursive multiple priors utility through a Bayes rule. See Epstein and Schneider (2003) for details about this property. In Hansen and Sargent (2006) there is a comprehensive discussion of the dynamic consistency issue under the robust control approach.

[8]For example, as ambiguity aversion translates into a higher effective risk aversion, it is a contribution for the explanation of the equity premium puzzle.

[9]For tractability reasons, the analysis is restricted to the class of Markov-Girsanov kernels. The absolute continuity assumption between P and P^h guarantees the equivalence property between the probability measures and, consequently, that the Cameron-Martin-Girsanov theorem can be applied. Moreover, from this theorem and considering the diffusion family of models under consideration, all that a probability measure change implies is the change of the drift function of the stochastic processes.

[10]In Trojani and Vanini (2004), p. 289, there is a detailed explanation supporting the rectangularity property of the present set of priors built under the constraint (9), and how this rectangular set of priors can be defined in the *k-ignorance* model of Chen and Epstein (2002).

Under an admissible contamination, P^h, the investment opportunity set is described by:

$$\begin{cases} \dfrac{dS_t}{S_t} & = \left(\mu + \sqrt{\dfrac{1}{y_t}}\rho h^y + \sqrt{\dfrac{1}{y_t}}\sqrt{1-\rho^2}h^\varepsilon\right)dt + \sqrt{\dfrac{1}{y_t}}\rho dW_y + \sqrt{\dfrac{1}{y_t}}\sqrt{1-\rho^2}dW_\varepsilon \\[4mm] dy_t & = \left[\kappa\left(\theta - y_t\right) + \sigma\sqrt{y_t}h^y\right]dt + \sigma\sqrt{y_t}dW_y \end{cases} \tag{10}$$

Note that in the "contaminated" investment opportunity set (10), the diffusion component continues to be driven by the same vector of independent Brownian motions as in (2)-(3).

With C_t, X_t and π_t representing the instantaneous consumption, wealth and fraction of wealth invested in the risky asset, wealth dynamics is given by:

$$dX_t = \pi_t X_t \frac{dS_t}{S_t} + (1-\pi_t)X_t r dt - C_t dt.$$

Considering the dynamics in (10), the intertemporal budget constraint faced by the ambiguous representative investor is given by:

$$dX_t = \left[\pi_t\left(\mu + \sqrt{\frac{1}{y_t}}\rho h^y + \sqrt{\frac{1}{y_t}}\sqrt{1-\rho^2}h^\varepsilon - r\right)X_t + rX_t - C_t\right]dt + \pi_t X_t \sqrt{\frac{1}{y_t}}\left(\rho dW_y + \sqrt{1-\rho^2}dW_\varepsilon\right). \tag{11}$$

The intertemporal optimization problem has a max-min structure. The investor chooses the consumption flow, $C : [t_0, +\infty[\to \mathbb{R}_+$, and the fraction of wealth to invest in the risky asset in each moment, $\pi : [t_0, +\infty[\to \mathbb{R}$, that maximize his expected utility (7). For each given choice, in the presence of multiple possible models, the ambiguity averse investor considers, from the set of alternative models, the worst-case scenario, i.e., the model that yields the lowest expected utility:

$$\sup_{\pi,C}\inf_{h\in\mathcal{H}} E_t^h\left[\int_t^\infty f\left(C_s, J_s\right)ds\right], \tag{12}$$

subject to the contaminated precision and wealth processes in (10) and (11), respectively.

The Bellman equation of this problem is:

$$\begin{aligned} 0 = \sup_{\pi,C}\inf_{h^y,h^\varepsilon} \Bigg\{ & f\left(C,J\right) + \left[\pi_t\left(\mu + \sqrt{\frac{1}{y_t}}\rho h^y + \sqrt{\frac{1}{y_t}}\sqrt{1-\rho^2}h^\varepsilon - r\right)X_t + rX_t - C_t\right]J_X + \\ & + \left[\kappa\left(\theta - y_t\right) + \sigma\sqrt{y_t}h^y\right]J_y + \frac{1}{2}\pi_t^2\frac{1}{y_t}X_t^2 J_{XX} + \frac{1}{2}\sigma^2 y_t J_{yy} + \pi_t X_t \rho\sigma J_{Xy}\Bigg\}, \end{aligned} \tag{13}$$

where $f\left(C,J\right)$ is the normalized aggregator given in (8) and J_X, J_y, J_{XX}, J_{yy} and J_{Xy} are partial derivatives of the value function $J\left(X_t, y_t\right)$.

Solving for the optimal vector (h^y, h^ε), i.e., for the worst-case contamination, and placing the result into (13), the Bellman equation of the problem becomes (Appendix 7.1):

$$
\begin{aligned}
0 \;=\; \sup_{\pi,C} \Bigg\{ & f\left(C,J\right) + \pi_t\left(\mu - r\right)X_t J_X + rX_t J_X - C_t J_X + \kappa\left(\theta - y_t\right)J_y + \frac{1}{2}\pi_t^2 \frac{1}{y_t}X_t^2 J_{XX} \\
& + \frac{1}{2}\sigma^2 y_t J_{yy} + \pi_t X_t \rho\sigma J_{Xy} - \sqrt{2\eta}\sqrt{\sigma^2 y_t J_y^2 + 2\sigma\rho\pi_t X_t J_y J_X + \pi_t^2 \frac{1}{y_t}X_t^2 J_X^2} \Bigg\} \, .
\end{aligned}
\tag{14}
$$

3 Optimal Consumption and Portfolio Rules

In general, obtaining closed-form solutions under stochastic investment opportunity sets is difficult. This difficulty is further enhanced by the presence of ambiguity. In this paper, we follow perturbation theory under robust control (e.g., Trojani and Vanini (2002)) to describe the solution of the problem under study. As in Trojani and Vanini (2004), we extend the asymptotic methods in Kogan and Uppal (2001) from models based on standard expected utility to models with ambiguity. This is allowed by the homotheticity of the robust control problem (10)-(12), which implies that the value function that solves the problem and the corresponding optimal consumption and portfolio policies are wealth scale-invariant.[11]

The rationale behind the perturbation (asymptotic) method is well described by Trojani and Vanini (2004): "[...] *formulate a general problem, find a particular relevant case that has a known solution, and use this as a starting point for computing the solution to nearby problems*". In our case, as in Trojani and Vanini (2004), the asymptotic solution of the problem under ambiguity will hold in neighborhoods of the model with no ambiguity.

The first step is to identify a set of parameters that parametrize the problem under study and specific parameter values for which the solution of the value function is known explicitly. Chacko and Viceira (2005) provided an exact solution for the case in which $\eta = 0$ (no ambiguity).

The value function that solves (14) for $\eta = 0$ is given by:[12]

$$
J\left(X_t, y_t\right) = exp\left\{g_0\left(y_t\right)\right\}\frac{X_t^{1-\gamma}}{1-\gamma} \, ,
\tag{15}
$$

where $g_0\left(y_t\right) = Ay_t + B$, with A and B given by

$$
A = \frac{\gamma\left(1-\gamma\right)\left\{\left[\frac{\beta+\kappa}{1-\gamma} - \frac{\rho\sigma(\mu-r)}{\gamma}\right] \pm \sqrt{\left[\frac{\beta+\kappa}{1-\gamma} - \frac{\rho\sigma(\mu-r)}{\gamma}\right]^2 - \frac{\sigma^2(\mu-r)^2\left[\gamma(1-\rho^2)+\rho^2\right]}{\gamma^2(1-\gamma)}}\right\}}{\sigma^2\left[\gamma\left(1-\rho^2\right)+\rho^2\right]} \, ,
\tag{16}
$$

$$
B = \left(1-\gamma\right)\left(ln\beta + \frac{r}{\beta} - 1\right) + \frac{\kappa\theta}{\beta}A \, .
\tag{17}
$$

The sign of the square-root in A is "+" for $\gamma > 1$ and "−" for $\gamma < 1$ (Appendix 7.2).

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[11] As explained in Trojani and Vanini (2002), studying non-homothetic robust control settings with perturbation methods is more difficult. Moreover, Maenhout (2004) points out some reasons to support the homotheticity assumption: "*Although economies exhibit growth, rates of return are stationary. Second, when the scale of the state variables matters, natural unit invariance of optimal decisions disappears and calibrations have to take this into account. Finally, homotheticity facilitates aggregation and the construction of a representative agent.*" As stated by Maenhout (2004), preserving homotheticity guarantees that "*[...] robustness will no longer wear off as wealth rises*".

[12] This expression is valid for $\gamma \neq 1$. If $\gamma = 1$, the value function is $J = ln\left(X_t\right)$.

Following the rationale above described, we will therefore perturb a benchmark economy in which $\eta = 0$, by considering small positive values of η. In order to obtain the asymptotic expansions of the optimal policies of the problem under study, we consider $J(X_t, y_t) = exp\left\{ g\left(y_t, \sqrt{2\eta}\right)\right\} \frac{X_t^{1-\gamma}}{1-\gamma}$ and the first-order expansion of $g\left(y_t, \sqrt{2\eta}\right)$ around $\eta = 0$:

$$g\left(y_t, \sqrt{2\eta}\right) = g_0(y_t) + g_1(y_t)\sqrt{2\eta} + \mathcal{O}_g^2\left(\sqrt{2\eta}\right), \tag{18}$$

where $\mathcal{O}_g^2\left(\sqrt{2\eta}\right)$ represents the residual of the first order expansion. As it is immediate from (18), $g_0(y_t)$ is the specification of $g\left(y_t, \sqrt{2\eta}\right)$ for the scenario when there is no ambiguity ($\eta = 0$).

Proposition 1 *Asymptotic optimal consumption and portfolio policies under ambiguity about the investment opportunity set dynamics (2)-(3), when $\gamma \geqslant \omega$, where $\omega = 1 - \frac{(\beta+\kappa)^2}{(\beta+\kappa)^2 + \sigma^2(\mu-r)^2 + 2\rho\sigma(\mu-r)(\beta+\kappa)}$, are given by:*[13]

$$C_t = \beta X_t + \mathcal{O}_c^2\left(\sqrt{2\eta}\right), \tag{19}$$

$$\pi_t = \frac{1}{\gamma + \sqrt{\frac{2\eta}{G_0(y_t)}}}\left[(\mu - r)\, y_t + \left(1 - \frac{1}{1-\gamma}\sqrt{\frac{2\eta}{G_0(y_t)}}\right)\sigma\rho A y_t\right] + \mathcal{O}_\pi^2\left(\sqrt{2\eta}\right), \tag{20}$$

with $G_0(y_t) = \left[\left(\frac{\mu-r+\rho\sigma A}{\gamma}\right)^2 + \left(\frac{\sigma A}{1-\gamma}\right)^2 + \frac{2\sigma\rho A(\mu-r+\rho\sigma A)}{(1-\gamma)\gamma}\right] y_t$ and A is given by (16).

Proof. *Appendix 7.3.*

The first comment on Proposition 1 is that the domain in which the solution is valid depends on the combination of the level of investor's risk aversion and on the characterization of the investment opportunity set dynamics (represented by ω). Note also that regarding the investor's preferences for the temporal resolution of risk, the domain of analysis ($\gamma \geqslant \omega$) includes scenarios where the investor: has preference for late resolution of risk ($\omega \leqslant \gamma < 1$); has preference for early resolution of risk ($\gamma > 1$); or is indifferent to that timing ($\gamma \to 1$). Only scenarios where the investor has a strong preference for late resolution of risk ($\gamma < \omega$) are excluded.

When there is no ambiguity, $\eta = 0$, the optimal consumption and portfolio rules are given by:

$$C_t = \beta X_t, \tag{21}$$

$$\pi_t = \frac{1}{\gamma}(\mu - r)\, y_t + \frac{\sigma\rho}{\gamma}A y_t, \tag{22}$$

which are the results in Chacko and Viceira (2005).

Comparing (19) and (21), it is clear that ambiguity has no first-order effect on the optimal consumption rule (which continues to be to consume a constant fraction β of current wealth). This means that the income and substitution effects on consumption that result from the change in the investment opportunity set exactly cancel out.

---

[13] For $\gamma < \omega$, the constant A in (16) is a complex number. Therefore, the value function (15) is only valid if $\gamma \geqslant \omega$.

On the other hand, from (20) and (22), it is immediate to conclude that ambiguity has a first-order impact on portfolio choice. There are novelties regarding existing results in the literature. First, the optimal allocation to the risky asset is instantaneously impacted by ambiguity, which contrasts with results in Faria et al. (2009). Secondly, the optimal portfolio rule is a non-linear function of y_t, which differs from the linear relationship that holds when there is no ambiguity (e.g., Chacko and Viceira (2005)) or when ambiguity is studied using RC with "multiplier preferences" (e.g., Maenhout (2004, 2006), Faria and Correia-da Silva (2010) and Xu et al. (2011)).

The structure of the optimal portfolio rule under ambiguity (20) continues to be the sum of two well-known components (Merton (1973)): (i) myopic demand, in this setting given by $\frac{\mu - r}{\gamma + \sqrt{\frac{2\eta}{G_0(y_t)}}} y_t$; and (ii) intertemporal hedging demand, given by $\frac{\left(1 - \frac{1}{1-\gamma}\sqrt{\frac{2\eta}{G_0(y_t)}}\right)\sigma\rho A}{\gamma + \sqrt{\frac{2\eta}{G_0(y_t)}}} y_t$. Comparing with the optimal portfolio rule without ambiguity (22), observe that the intertemporal hedging demand vanishes (and, therefore, the myopic demand becomes optimal) as: the coefficient of relative risk aversion tends to 1 ($\gamma \to 1$); investment opportunities are constant ($\sigma = 0$) or, being time-varying, it is not possible to use the risky asset to hedge against those changes ($\rho = 0$). Notice also that the ratio between myopic and intertemporal hedging demand is a function of instantaneous precision (y_t), contrarily to what happens when there is no ambiguity ($\eta = 0$).

Additionally, without ambiguity, an investor with $\gamma > 1$ has a negative intertemporal hedging demand, and the opposite when $\omega \leqslant \gamma < 1$, which is consistent with the findings in Chacko and Viceira (2005). When risk aversion is low ($\omega \leqslant \gamma < 1$), the investor is ready to support a worse performance when precision is low for extra performance when precision is high (recall that $\rho > 0$). An investor with high risk aversion ($\gamma > 1$) is not willing to accept this trade-off. With the introduction of ambiguity, this relation is not so trivial: investors with low risk aversion ($\gamma < 1$) that face a sufficiently high level of ambiguity (high η), have a negative intertemporal hedging demand.[14]

Note that the myopic (M) and the intertemporal hedging (H) demand can be written as:

$$\pi_t^M(\eta) = \pi_t^M(0) \frac{\gamma}{\gamma + \sqrt{\frac{2\eta}{G_0(y_t)}}}, \tag{23}$$

$$\pi_t^H(\eta) = \pi_t^H(0) \frac{\gamma\left(1 - \frac{1}{1-\gamma}\sqrt{\frac{2\eta}{G_0(y_t)}}\right)}{\gamma + \sqrt{\frac{2\eta}{G_0(y_t)}}}, \tag{24}$$

where $\pi_t^M(0)$ and $\pi_t^H(0)$ represent the myopic and intertemporal hedging demand components without ambiguity aversion. Both ratios $\pi_t^M(\eta)/\pi_t^M(0)$ and $\pi_t^H(\eta)/\pi_t^H(0)$ depend on y_t through $G_0(y_t)$. It is clear that $G_0(y_t) > 0$ which, from (23) and (24), implies that the reduction in the optimal risky asset demand is a positive function of the level of ambiguity of the representative investor (higher η). The result that ambiguity aversion reduces the demand for the risky asset is the standard result within the still recent literature on portfolio choice under ambiguity. We extend this result to a setting where stochastic precision is one of the sources of ambiguity, in a "constraint preferences" RC setting.

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[14]For this to happen when $\omega \leqslant \gamma < 1$, it is necessary that $\gamma > 1 - \sqrt{\frac{2\eta}{G_0(y_t)}}$.

4   Alternative Scenarios

In section 4.1, asymptotically optimal consumption and portfolio rules are derived for the case in which ambiguity is exclusively about the stochastic process that generates the return of the risky asset. In section 4.2, the same is done in a scenario where ambiguity is only about the precision process.

4.1   Ambiguity exclusively about the return process

Consider the investment opportunity set described in section 2.1, and the existence of ambiguity exclusively about the return process. Formally, restrict the possible perturbations to be of the form $h = \begin{bmatrix} 0 & h^\varepsilon \end{bmatrix}^\top$. From (13), the corresponding Bellman equation is given by:

$$
\begin{aligned}
0 \quad = \quad & \sup_{\pi, C} \inf_{h^\varepsilon} \left\{ f\left(C, J\right) + \left[\pi_t \left(\mu + \sqrt{\frac{1}{y_t}} \sqrt{1 - \rho^2} h^\varepsilon - r \right) X_t + r X_t - C_t \right] J_X + \right. \\
& \left. + \kappa \left(\theta - y_t \right) J_y + \frac{1}{2} \pi_t^2 \frac{1}{y_t} X_t^2 J_{XX} + \frac{1}{2} \sigma^2 y_t J_{yy} + \pi_t X_t \rho \sigma J_{Xy} \right\} .
\end{aligned}
\tag{25}
$$

Observe that the objective function is monotonically increasing in h^ε. This implies, from (9), that the worst-case contamination is the corner solution $h^\varepsilon = -\sqrt{2\eta}$. Introducing this result into (25), the Bellman equation of the problem when $h = \begin{bmatrix} 0 & -\sqrt{2\eta} \end{bmatrix}^\top$ becomes:

$$
\begin{aligned}
0 \quad = \quad & \sup_{\pi, C} \left\{ f\left(C, J\right) + \left[\pi_t \left(\mu - \sqrt{\frac{1}{y_t}} \sqrt{1 - \rho^2} \sqrt{2\eta} - r \right) X_t + r X_t - C_t \right] J_X + \right. \\
& \left. + \kappa \left(\theta - y_t \right) J_y + \frac{1}{2} \pi_t^2 \frac{1}{y_t} X_t^2 J_{XX} + \frac{1}{2} \sigma^2 y_t J_{yy} + \pi_t X_t \rho \sigma J_{Xy} \right\} .
\end{aligned}
\tag{26}
$$

Following the same reasoning as in section 3, the optimal portfolio and consumption rules are deducted for the particular case of ambiguity exclusively about the return process.

Proposition 2 *Asymptotic optimal consumption and portfolio policies under ambiguity about the risky asset return process (2), when $\gamma \geqslant \omega$, where $\omega = 1 - \frac{(\beta + \kappa)^2}{(\beta + \kappa)^2 + \sigma^2 (\mu - r)^2 + 2\rho\sigma(\mu - r)(\beta + \kappa)}$, are:*

$$
C_t \quad = \quad \beta X_t + \mathcal{O}_c^2 \left(\sqrt{2\eta} \right),
\tag{27}
$$

$$
\pi_t \quad = \quad \frac{1}{\gamma} \left(\mu - r \right) y_t - \frac{\sqrt{1 - \rho^2} \sqrt{2\eta}}{\gamma} \sqrt{y_t} + \frac{\sigma \rho}{\gamma} A y_t + \mathcal{O}_\pi^2 \left(\sqrt{2\eta} \right),
\tag{28}
$$

with A given by (16).
Proof. *Appendix 7.3.*

The main conclusions from Proposition 1 extend to Proposition 2. The main difference is that now the effect of ambiguity on portfolio choice only concerns the myopic demand, as the intertemporal hedging demand remains equal to $\frac{\sigma \rho}{\gamma} A y_t$. This is natural because the intertemporal hedging demand

is driven by the dynamics of stochastic variance. It is, therefore, unaffected by ambiguity about the return process.

4.2   Ambiguity exclusively about the precision process

To investigate the case in which ambiguity is only about the precision process, it is not appropriate to consider the investment opportunity set described by (2)-(3). In that case, a contamination of the precision process would be transmitted to the return process through the assumed correlation between return and precision. The appropriate setting is the following:

$$
\begin{aligned}
\frac{dS_t}{S_t} &= \mu dt + \sqrt{\frac{1}{y_t}} dW_S, & (29)\\
dy_t &= \kappa \left(\theta - y_t\right) dt + \sigma \sqrt{y_t} \left(\rho dW_S + \sqrt{1 - \rho^2} dW_\varepsilon\right). & (30)
\end{aligned}
$$

Ambiguity is again introduced through Markovian contaminating drift processes. If we allowed any contamination $h = \begin{bmatrix} h^s & h^\varepsilon \end{bmatrix}^\top$ satisfying the entropy bound (9), then we would obtain exactly the same Bellman equation as in section 2.3 (Appendix 7.4). Therefore, as the Bellman equation would still be given by (14), Proposition 1 would remain valid for this alternative setting. Since now we want to study the case in which ambiguity is exclusively about the precision process, we will restrict the contaminations to the precision process, i.e., we will consider $h = \begin{bmatrix} 0 & h^\varepsilon \end{bmatrix}^\top$. The investment opportunity set is now described by:

$$
\begin{cases}
\frac{dS_t}{S_t} &= \mu dt + \sqrt{\frac{1}{y_t}} dW_S, \\
dy_t &= \left[\kappa \left(\theta - y_t\right) + \sigma \sqrt{y_t \left(1 - \rho^2\right)} h^\varepsilon\right] dt + \sigma \sqrt{y_t} \left(\rho dW_S + \sqrt{1 - \rho^2} dW_\varepsilon\right),
\end{cases} \quad (31)
$$

and the corresponding intertemporal budget constraint faced by the ambiguous representative investor is given by:

$$
dX_t = \left[\pi_t \left(\mu - r\right) X_t + rX_t - C_t\right] dt + \pi_t \sqrt{\frac{1}{y_t}} X_t dW_S. \quad (32)
$$

The Bellman equation of the optimization problem under this setting is:

$$
\begin{aligned}
0 &= \sup_{\pi,C} \inf_{h^s, h^\varepsilon} \left\{ f\left(C, J\right) + \left[\pi_t \left(\mu - r\right) X_t + rX_t - C_t\right] J_X + \frac{1}{2} \pi_t^2 \frac{1}{y_t} X_t^2 J_{XX} + \right. \\
&\quad \left. + \left[\kappa \left(\theta - y_t\right) + \sigma \sqrt{y_t \left(1 - \rho^2\right)} h^\varepsilon\right] J_y + \frac{1}{2} \sigma^2 y_t J_{yy} + \pi_t X_t \rho \sigma J_{Xy} \right\}.
\end{aligned} \quad (33)
$$

With $J_y \geq 0$, which holds at least for small ambiguity levels, the worst-case contamination is the corner solution $h^\varepsilon = -\sqrt{2\eta}$. Independently of whether we consider $h^\varepsilon = -\sqrt{2\eta}$ or $h^\varepsilon = \sqrt{2\eta}$, the optimal consumption and portfolio rules are as follows.

Proposition 3 *Asymptotic optimal consumption and portfolio policies under ambiguity about the*

precision process (30), when $\gamma \geqslant \omega$, where $\omega = 1 - \frac{(\beta+\kappa)^2}{(\beta+\kappa)^2 + \sigma^2(\mu-r)^2 + 2\rho\sigma(\mu-r)(\beta+\kappa)}$, are given by:

$$C_t \;\; = \;\; \beta X_t + \mathcal{O}_c^2 \left(\sqrt{2\eta} \right), \tag{34}$$

$$\pi_t \;\; = \;\; \frac{1}{\gamma} \left(\mu - r \right) y_t + \frac{\sigma\rho}{\gamma} A y_t + \mathcal{O}_\pi^2 \left(\sqrt{2\eta} \right), \tag{35}$$

with A given by (16).

The main conclusion from Proposition 3 is that ambiguity about the precision process has no first-order impact on optimal consumption and portfolio choice.

Overall, optimal portfolio rules were deducted for the scenarios where there exists ambiguity: about the dynamics of both the return and its precision (Proposition 1), only about the dynamics of the return (Proposition 2); and only about the dynamics of precision (Proposition 3). The conclusion is that, from a theoretical point of view, the relevant channel through which ambiguity impacts portfolio decisions is the return of the risky asset. Ambiguity about the precision process is only relevant if, through the assumed correlation between return shocks and precision shocks, it also induces ambiguity about the return process: this happens under the setting (2)-(3), but not under the setting (29)-(30). In the next section we evaluate the empirical relevance of those findings.

5 Simulation

Chacko and Viceira (2005) found that, calibrating their model to long-run U.S. data, the optimal intertemporal hedging demand is empirically small. The same conclusion was reached by Liu (2007). This suggests that the "risk dimension" of stochastic variance is empirically not very relevant to dynamic portfolio choice. However, Chacko and Viceira (2005), in their concluding remarks, acknowledged that an important caveat of their analysis is that they have counterfactually assumed that investors observe variance and take as true the empirical estimates of the parameters of the variance process.

Following this lead, we have generalized their model to account for ambiguity about the stochastic investment opportunity set. As a result, the "myopic demand" and the "intertemporal hedging demand" became ambiguity-adjusted.

Our simulation suggests that the ambiguity impact on the allocation to the risky asset has a relevant empirical dimension. However, this effect is essentially due to ambiguity about the return process. The impact of ambiguity about the variance process is empirically very low.

The reference parameter values used in the simulation are those estimated by Chacko and Viceira (2005), based on monthly excess stock returns on the CRSP value-weighted portfolio over the T-Bill

rate from January 1926 through December 2000:

$$
\begin{aligned}
\mu - r &= 0.0811, \\
\kappa &= 0.3374, \\
\theta &= 27.9345, \\
\sigma &= 0.6503, \\
\rho &= 0.5241, \\
r &= 0.015, \\
\beta &= 0.06.
\end{aligned}
\tag{36}
$$

From (5), the expected standard deviation of returns is 19.1314%.

Implications of ambiguity on the optimal allocation to the risky asset (Proposition 1) are exemplified in Table 1. The first column presents results for the scenario without ambiguity. The other three columns represent scenarios with three arbitrary levels of ambiguity: $\eta = 0.005, 0.01, 0.02$. Recall that alternative models have to be statistically close so that the investor is ambiguous about the reference model. This implies small η values. In Trojani and Vanini (2004), two arbitrary values for η are used $(0.005, 0.01)$ while the value implied by all calibrations in Gagliardini et al. (2009) is lower than 0.0136.[15]

Simulations are run for different levels of risk aversion ($\gamma = 0.75, 1, 2, 4, 20, 40$), assuming unit elasticity of intertemporal substitution ($\psi = 1$). In panel A, we show the mean allocation to the risky asset (percentage of wealth). In panel B, the intertemporal hedging demand is shown as a percentage of the myopic demand. In panels C and D, the ambiguity effect is explicitly calculated as a percentage of total risky asset demand and myopic demand.

---

[15]Those values for η can be taken as a reference without introducing any kind of bias in our analysis. This is because, as explained in section 2.3, the diffusion dimension of "contaminated" processes is unchanged vs. non-contaminated processes (only the drift functions are affected).

Table 1: Ambiguity impact on optimal risky asset demand (Proposition 1).

	η			
	0	0.005	0.01	0.02
	A - Mean allocation to risky asset (%)			
γ				
0.75	305,66	245,80	227,16	204,98
1.00	226,55	180,98	166,91	150,24
2.00	111,37	89,59	82,80	74,73
4.00	55,24	44,43	41,07	37,07
20.00	10,98	8,83	8,16	7,37
40.00	5,48	4,41	4,08	3,68
	B - Ratio of hedging demand over myopic demand (%)			
γ				
0.75	1,19	0,36	0,01	-0,48
1.00	0,00	-0,84	-1,19	-1,68
2.00	-1,68	-2,46	-2,78	-3,24
4.00	-2,47	-3,24	-3,56	-4,01
20.00	-3,09	-3,85	-4,16	-4,61
40.00	-3,17	-3,92	-4,24	-4,68
	C - Reduction in allocation to risky asset due to ambiguity (%)			
γ				
0.75	0,00	-19,58	-25,68	-32,94
1.00	0,00	-19,57	-25,67	-32,92
2.00	0,00	-19,56	-25,65	-32,90
4.00	0,00	-19,56	-25,65	-32,89
20.00	0,00	-19,55	-25,64	-32,88
40.00	0,00	-19,55	-25,64	-32,88
	D - Reduction of myopic demand due to ambiguity (%)			
γ				
0.75	0,00	-18,92	-24,81	-31,81
1.00	0,00	-18,92	-24,81	-31,81
2.00	0,00	-18,92	-24,81	-31,81
4.00	0,00	-18,92	-24,81	-31,81
20.00	0,00	-18,92	-24,81	-31,81
40.00	0,00	-18,92	-24,81	-31,81

Note 1 - **Panel A:** $\pi_\theta = E[\pi_t] \times 100$, with $E(y_t) = \theta$; **Panel B:** $\frac{\left(1 - \frac{1}{1-\gamma}\sqrt{\frac{2\eta}{G_0(E(y_t))}}\right)\sigma\rho A}{(\mu-r)} \times 100$, with $E(y_t) = \theta$; **Panel C:** $\left[\frac{(20)}{(22)} - 1\right] \times 100$, with $E(y_t) = \theta$; **Panel D:** $\left[\frac{\gamma}{\gamma + \sqrt{\frac{2\eta}{G_0(E(y_t))}}} - 1\right] \times 100$, with $E(y_t) = \theta$;

Note 2 - $\omega = 0.14 < 0.75$: domain of Proposition 1 is guarantied.

Results presented in Table 1 are consistent with comments in section 3 and show that ambiguity is empirically relevant: even for a low level of ambiguity (second column in Table 1), ambiguity implies a 20% decrease of the mean optimal demand of the risky asset (panel C). Consider, for example, a risk-averse investor, with $\gamma = 2$, that is ambiguity-neutral. His mean optimal allocation to the risky asset corresponds to 111.4% of his wealth. If this investor becomes ambiguity averse, for example, with $\eta = 0.01$, his mean optimal allocation to the risky asset declines to 82,8% of his wealth. These findings can contribute to the explanation of the so called "flight to quality" effect (stylized fact in financial markets): when investors, for some reason, become more "nervous" and uncertain about market conditions, they reduce exposure to risky assets and invest in less risky or riskless assets.

Panel A in Table 1 shows that the demand for the risky asset is decreasing with risk aversion, γ, and with the level of ambiguity, η. Since the long-term expected return on wealth is measured by $\pi_\theta(\mu - r) + r$, it is a decreasing function of both risk aversion and the level of ambiguity.

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Figure 1: Long-term expected return on wealth as a function of ambiguity and risk aversion.
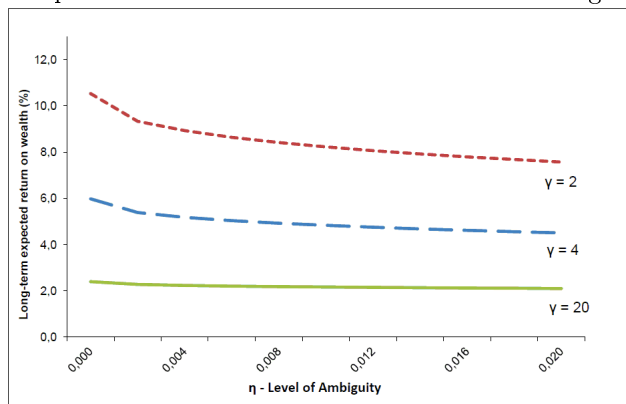
Moreover, the higher the level of ambiguity and the level of instantaneous volatility (inverse of precision) the higher is the impact from ambiguity. This is represented in Figure 2, where it is also evident that the ambiguity effect through the relevant support of precision (or variance) has a non-linear nature.

Figure 2: Ambiguity effect as a function of instantaneous volatility
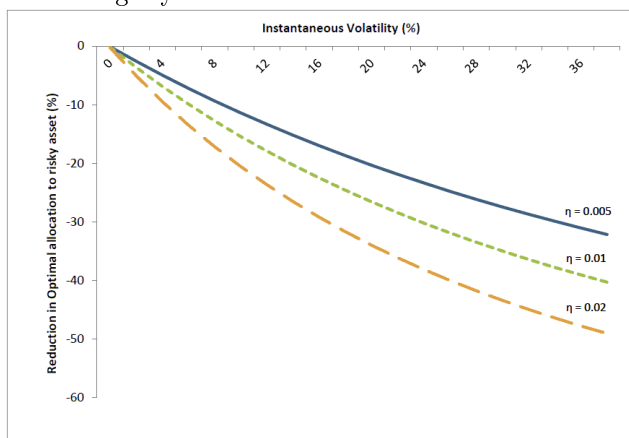
Note 1 - Volatility understood as variance v_t, computed from the level of precision y_t using (5);
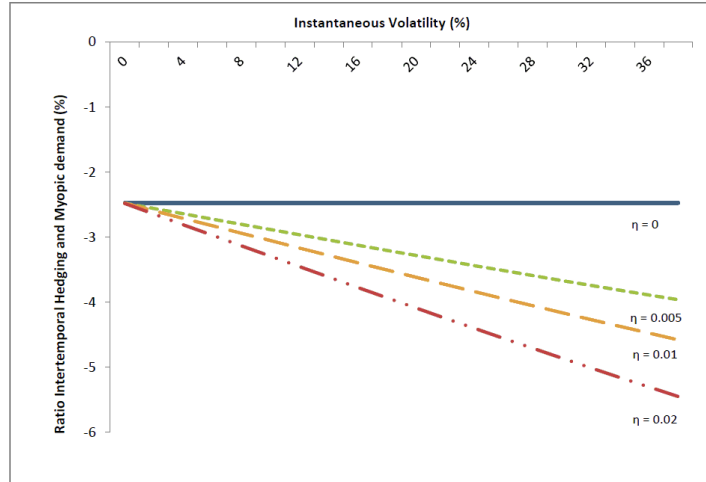***Note 2 -*** Simulation with $\gamma = 4$.

These findings suggest that the intensity, i.e., the speed and depth, of the asset reallocation implied in the "flight to quality" phenomena should increase with the level of ambiguity and of instantaneous volatility. This is intuitive: in an anxious market environment as the one following Lehman Brothers collapse in September 2008 (which Blanchard (2009) suggestively named as "Knight time"), during which the VIX index[16] reached its historical maximum of 80.86% (20th November, 2008), the speed and volumes of risky asset "sell-off" trades were much higher than in stable market conditions.

Additionally, as pointed out in section 3, the ratio between intertemporal hedging demand and

---

[16]The VIX Index from CBOE is probably the most used volatility index, both in the literature and in the industry. It measures the one-month implied volatility in the S&P 500 Index option prices. For full details on the VIX Index construction methodology please see http://www.cboe.com/micro/vix/.

myopic demand becomes a function of precision when ambiguity is considered. This has an intuitive interpretation: the higher the level of instantaneous volatility, everything else constant, the more the investor is concerned about the impact from volatility changes in his intertemporal utility and, being ambiguous about the process that drives volatility, the higher is the optimal hedging demand. This is graphically highlighted in Figure 3, where it is also clear that the dimension of this adjustment is empirically small.

Figure 3: Hedging Demand vs Myopic Demand as a function of instantaneous volatility

Note 1 - Volatility understood as variance v_t, computed from the level of precision y_t using (5);
Note 2 - Simulation with $\gamma = 4$.

Panel B reports estimates of the intertemporal hedging demand, measured as a ratio of myopic demand. Again, results show that ambiguity reinforces the effect of risk aversion: the higher the ambiguity and risk aversion the higher the relative importance of the intertemporal hedging demand. However, the intertemporal hedging demand is always small - even for a highly risk and ambiguity averse investor ($\gamma = 40$, $\eta = 0.02$). The novelty with ambiguity is that for low risk averse investors that face a high level of ambiguity (e.g., $\gamma = 0.75$, $\eta = 0.02$), the intertemporal hedging demand becomes negative. This confirms the predictions highlighted in section 3. With no ambiguity or moderate levels of ambiguity, the intertemporal hedging demand is positive when $\gamma < 1$ and negative when $\gamma \geqslant 1$.

The fact that intertemporal hedging demand is empirically small, even for higher levels of ambiguity, means that ambiguity impacts optimal portfolio decision essentially through the myopic component of demand. This is confirmed by the results disclosed in Panels C and D. This has a clear economic meaning and provides an answer to the research question addressed in this paper: even accounting for ambiguity about the stochastic volatility process, it is found that the optimal hedging demand required by investors for protection against adverse changes in volatility is still very low. Investors are essentially focused in the short term risk-return characteristics of the risky asset (myopic dimension), and stochastic volatility has low relevance for optimal intertemporal portfolio decisions: this has been found under settings were uncertainty is exclusively risk (for example in Chacko and Viceira (2005) and Liu (2007)) and we extend the conclusion for a setting where uncertainty also has an ambiguity dimension.

Moreover, recalling conclusions from section 4.2, this setting (Table 1) is the only one where ambiguity about stochastic volatility process impacts optimal portfolio decisions.

At last, as a "cross-check" test, the scenario with ambiguity exclusively about the risky asset return process (Proposition 2) was simulated, confirming its empirical relevance.

Table 2: Ambiguity impact on optimal risky asset demand (Proposition 2).

	η			
	0	0.005	0.01	0.02
	A - Mean allocation to risky asset (%)			
γ				
0.75	305,66	245,64	220,78	185,62
1.00	226,55	181,54	162,89	136,52
2.00	111,37	88,87	79,55	66,36
4.00	55,24	43,98	39,32	32,73
20.00	10,98	8,73	7,79	6,48
40.00	5,48	4,36	3,89	3,23
	B - Reduction in allocation to risky asset due to ambiguity (%)			
γ				
0.75	0,00	-19,64	-27,77	-39,27
1.00	0,00	-19,87	-28,10	-39,74
2.00	0,00	-20,21	-28,58	-40,42
4.00	0,00	-20,37	-28,81	-40,75
20.00	0,00	-20,50	-28,99	-41,01
40.00	0,00	-20,52	-29,02	-41,04

Note 1 - **Panel A:** $\pi_\theta = E\left[\pi_t\right] \times 100$, with $E\left(y_t\right) = \theta$; **Panel B:** $\left[\frac{(28)}{(22)} - 1\right] \times 100$, with $E\left(y_t\right) = \theta$;
Note 2 - $\omega = 0.14 < 0.75$: domain of Proposition 2 is guarantied.

The comparison of results in Panel B of Table 2 with those in Panel C of Table 1 leads to an important finding: when ambiguity is concentrated in the return process (Table 2), its impact on the optimal demand for the risky asset is higher.

Our conclusion that uncertainty about stochastic volatility has low impact on portfolio decisions may not be robust to the introduction of multiple risky assets (multivariate stochastic variance and covariance setting). In such a setting, Buraschi et al. (2010) showed that joint features of volatility and correlation dynamics play an important role in optimal portfolios. For example, they estimate that, in a univariate stochastic volatility model, total hedging demand for S&P 500 futures of investors with $\gamma = 8$ and investment horizon of 10 years is 4.8% of the myopic demand. This is consistent with our results in Table 1. However, in a model with three risky assets, the estimated total hedging demand for S&P500 futures jumps to 28% of myopic demand, with 11% and 17% of volatility and correlation hedging demand respectively.

Interestingly, Buraschi et al. (2010) also find that the optimal hedging demand against correlation risk typically dominates hedging against volatility risk: this is, at least partially, explained by the higher persistence of correlation risk versus that of volatility risk. In a study about the relation between correlation risk and the cross-section of hedge fund returns, Buraschi et al. (2012) find evidence that correlation risk is the most significant risk factor for the explanation of hedge fund returns. Those findings suggest that correlation, more than volatility, is the relevant uncertainty factor to be "controlled" and therefore commanding investors intertemporal hedging demand.

6   Concluding Remarks

We study optimal dynamic consumption and portfolio choice with stochastic variance, by introducing ambiguity about the stochastic processes that generate the dynamic investment opportunity set.

Long-horizon investors with recursive preferences, as defined by Duffie and Epstein (1992b) with Kreps and Porteus' (1978) specification, have two assets to invest in, a risk-free asset and a risky asset. The investor considers a reference model for the data generating processes but, not being totally sure about it, takes into account a set of statistically close models (with the relative entropy between models being bounded). Ambiguity aversion in the spirit of Gilboa and Schmeidler (1989) implies that investor will consider the worst possible alternative model, i.e., the one associated with the lowest expected utility. Optimal dynamic policies under ambiguity are deducted by making use of perturbation theory techniques for robust control problems.

The main conclusions of this paper concern the impact on optimal dynamic policies from ambiguity about the data generating processes, both when ambiguity is simultaneously about the return and volatility processes and when it is exclusively about one of them. In all scenarios, the optimal consumption policy is to consume a constant fraction of wealth. It is found that ambiguity does not impact the optimal consumption-wealth ratio, at least until a first order approximation with respect to the level of ambiguity.

Conversely, ambiguity about the data generating processes reduces the optimal demand for the risky asset, with that effect being non-uniform in the variance domain. The same happens when there is ambiguity only about the risky asset return. When ambiguity is exclusively about the stochastic variance process it is found that there is no impact on the optimal portfolio rule. The conclusion is that ambiguity about the stochastic variance process is only relevant as long as, through a specific correlation structure, it also induces ambiguity about the return stochastic process.

Making use of long-run US data, we measure the empirical dimensions of those effects. The first conclusion is that ambiguity about the stochastic processes driving the investment opportunity set is empirically relevant for portfolio decisions. This can be a contribute for the explanation of the "fly to quality" stylized fact in financial markets. Our simulation suggests that this highly relevant ambiguity effect on the risky asset demand acts mainly through the myopic component. The first implication is the confirmation that, under our setting and simulation, ambiguity about the risky asset return process is empirically much more relevant than ambiguity about stochastic volatility process. The second implication is that, even accounting for ambiguity about the stochastic volatility process, it is found that the optimal hedging demand required by investors for protection against adverse changes in volatility is still very low. Investors are essentially focused in short term risk-return characteristics of the risky asset (myopic dimension), and stochastic volatility has low relevance for intertemporal portfolio decisions: this has been found under settings were uncertainty is exclusively risk (e.g. in Chacko and Viceira (2005) and Liu (2007)) and we extend that conclusion for a setting where uncertainty also has an ambiguity dimension.

Our conclusion that uncertainty about stochastic volatility has low impact on portfolio decisions may change significantly if a multivariate stochastic variance setting (multiple risky assets) is considered, as Buraschi et al. (2010) shows: the authors find that hedging demands are typically four to five times higher than those of models with constant correlations or single-factor stochastic volatility.

Moreover, Buraschi et al. (2010) also find that correlation risk hedging demand is typically higher than volatility risk demand suggesting that correlation, more than volatility, is the crucial uncertainty factor to hedge in an intertemporal optimization portfolio problem.

An interesting research topic for future work is therefore the consideration of ambiguity about stochastic variance-covariance dynamics in a multivariate model for optimal intertemporal portfolio choice.

7 Appendices

7.1 Bellman Equation (14)

Define Λ as the diffusion matrix of state variables y_t and X_t, according to processes in (10) and (11):

$$
\Lambda \;=\; \begin{bmatrix} \sigma\sqrt{y_t} & 0 \\[2ex] \rho\pi_t\sqrt{\frac{1}{y_t}}X_t & \pi_t\sqrt{\frac{1}{y_t}}X_t\sqrt{1-\rho^2} \end{bmatrix}. \tag{37}
$$

Minimization of (13) with respect to the vector h gives (see, for e.g., Anderson et al. (1998) p. 22):

$$
\begin{bmatrix} h^y \\[2ex] h^\varepsilon \end{bmatrix} \;=\; -\frac{\sqrt{2\eta}}{\sqrt{\begin{bmatrix} J_y & J_X \end{bmatrix}\Lambda\Lambda^\top\begin{bmatrix} J_y & J_X \end{bmatrix}^\top}}\Lambda^\top\begin{bmatrix} J_y \\[2ex] J_X \end{bmatrix}. \tag{38}
$$

Replacing (37) in (38), the vector of optimal contaminating drifts is obtained and given by:

$$
\begin{bmatrix} h^y \\[2ex] h^\varepsilon \end{bmatrix} \;=\; -\frac{\sqrt{2\eta}}{\sqrt{J_y^2\sigma^2 y_t + 2J_yJ_X\sigma\rho\pi_t X_t + J_X^2\pi_t^2\frac{1}{y_t}X_t^2}}\begin{bmatrix} \sigma\sqrt{y_t}J_y + \rho\pi_t\sqrt{\frac{1}{y_t}}X_tJ_X \\[2ex] \pi_t\sqrt{\frac{1}{y_t}}X_t\sqrt{1-\rho^2}J_X \end{bmatrix}.
$$

Substituting this result into (13), after some algebra, gives (14).

$\square$

7.2 Sign of the square-root in (16)

Since $\psi = 1$, as $\gamma \to 1$, the utility representation (8) converges to the log-utility representation. The optimal portfolio rule without ambiguity ($\eta = 0$), given by (22), in the special case of log-utility ($\gamma = \psi = 1$) is well-known (Merton (1969, 1971, 1973)):

$$
\pi_t = (\mu - r)\,y_t\,,
$$

i.e., the intertemporal hedging demand component disappears (if $\psi = \gamma = 1$, then $A = B = 0$). It is therefore necessary to guarantee that $\lim_{\gamma\to 1} A = 0$, which implies that $\lim_{\gamma\to 1} B = 0$. The limit of (22) as

$\gamma \to 1$ is:

$$\lim_{\gamma \to 1} \pi_t = (\mu - r) \, y_t + \left(\lim_{\gamma \to 1} A \right) \rho \sigma y_t \,.$$

From (16), $\lim_{\gamma \to 1} A$ is:

$$\lim_{\gamma \to 1} A = \quad \frac{(\beta + \kappa) \pm \lim_{\gamma \to 1} (1 - \gamma) \gamma \sqrt{\left[\frac{\rho \sigma (\mu - r)}{\gamma} - \frac{\beta + \kappa}{1 - \gamma} \right]^2 - \frac{\sigma^2 (\mu - r)^2 \left[\gamma \left(1 - \rho^2 \right) + \rho^2 \right]}{\gamma^2 (1 - \gamma)}}}{\sigma^2} \,. \tag{39}$$

If $\gamma \to 1^+$, i.e., $\gamma > 1$, then $(1 - \gamma) < 0$ and the discriminant of the square root in (39) is always > 0. By assumption, $\beta + \kappa > 0$, which implies that, in order to have $\lim_{\gamma \to 1^+} A = 0$, the "+" sign must be considered.

The same rationale implies that when $\gamma < 1$, the "-" sign of the square root guarantees that $\lim_{\gamma \to 1^-} A = 0$ (it can be easily shown that the discriminant of the square root in (39) is positive as γ approaches 1 from below).

$\square$

7.3   Optimal Consumption and Portfolio rules

7.3.1   Domain $\gamma \geqslant \omega$

The domain of analysis is set so that A in (16) is a real number, i.e., its discriminant is non-negative. Consequently the condition to be satisfied is

$$\left[\frac{\rho \sigma (\mu - r)}{\gamma} - \frac{\beta + \kappa}{1 - \gamma} \right]^2 \quad \geqslant \quad \frac{\sigma^2 (\mu - r)^2 \left[\gamma \left(1 - \rho^2 \right) + \rho^2 \right]}{\gamma^2 (1 - \gamma)} \,. \tag{40}$$

For $\gamma > 1$ it is straightforward to conclude that $\left[\frac{\rho \sigma (\mu - r)}{\gamma} - \frac{\beta + \kappa}{1 - \gamma} \right]^2 > \frac{\sigma^2 (\mu - r)^2 \left[\gamma \left(1 - \rho^2 \right) + \rho^2 \right]}{\gamma^2 (1 - \gamma)}$, and therefore (40) is always true.

For $\gamma < 1$, (40) is true as long as:

$$\frac{\gamma}{1 - \gamma} \quad \geqslant \quad \frac{\sigma^2 (\mu - r)^2}{(\beta + \kappa)^2} + \frac{2 \rho \sigma (\mu - r)}{(\beta + \kappa)}$$

$$\Leftrightarrow \gamma \quad \geqslant \quad \frac{\sigma^2 (\mu - r)^2 + 2 \rho \sigma (\mu - r) (\beta + \kappa)}{(\beta + \kappa)^2 + \sigma^2 (\mu - r)^2 + 2 \rho \sigma (\mu - r) (\beta + \kappa)}$$

$$\Leftrightarrow \gamma \quad \geqslant \quad \omega \,,$$

where $\omega = 1 - \frac{(\beta + \kappa)^2}{(\beta + \kappa)^2 + \sigma^2 (\mu - r)^2 + 2 \rho \sigma (\mu - r) (\beta + \kappa)}$. Note that $\omega < 1$.

$\square$

7.3.2 Optimal rules (19) and (20)

Considering the Bellman equation (14) and a value function with the form $J(X_t, y_t) = exp\left\{g\left(y_t, \sqrt{2\eta}\right)\right\}\frac{X_t^{1-\gamma}}{1-\gamma}$, the FOC with respect to π_t gives:

$$\pi_t = \frac{1}{\gamma + \sqrt{\frac{2\eta}{G(\pi_t, y_t)}}}\left[(\mu - r)y_t + \left(1 - \frac{1}{1-\gamma}\sqrt{\frac{2\eta}{G(\pi_t, y_t)}}\right)\sigma\rho y_t \frac{\partial g}{\partial y}\right], \qquad (41)$$

where:

$$G(\pi_t, y_t) = \frac{\pi_t^2}{y_t} + \frac{2\sigma\rho\pi_t\frac{\partial g}{\partial y_t}}{1-\gamma} + \frac{\sigma^2 y_t\left(\frac{\partial g}{\partial y_t}\right)^2}{(1-\gamma)^2}, \qquad (42)$$

i.e., optimal portfolio rule under ambiguity is the solution of an implicit function in π_t. In order to provide an approximate solution for this optimization problem, consider the first order expansions in $\sqrt{2\eta}$ of the functions g and π_t. Expansion of g is given in (18) and expansion of π_t is given by:

$$\pi\left(y_t, \sqrt{2\eta}\right) = \pi_0(y_t) + \pi_1(y_t)\sqrt{2\eta} + \mathcal{O}_\pi^2\left(\sqrt{2\eta}\right), \qquad (43)$$

where $\mathcal{O}_\pi^2\left(\sqrt{2\eta}\right)$ is a symbol representing terms of higher order in $\sqrt{2\eta}$. $\pi_0(y_t)$ represents the solution when there is no ambiguity ($\eta = 0$), being given by (22). With no ambiguity, the value of $G(\pi_t, y_t)$ is:

$$G_0(y_t) = \left[\left(\frac{\mu - r + \rho\sigma A}{\gamma}\right)^2 + \left(\frac{\sigma A}{1-\gamma}\right)^2 + \frac{2\rho\sigma A(\mu - r + \rho\sigma A)}{(1-\gamma)\gamma}\right]y_t.$$

To find $\pi_1(y_t)$, since we are neglecting terms of higher order in $\sqrt{2\eta}$, we can consider the approximation $G_0(y_t)$ instead of $G(\pi_t, y_t)$. Therefore, the approximate optimal porfolio choice can be written as in (20).

Regarding the optimal consumption rule (19), computations are more straightforward. Considering the Bellman (14) and the aggregator (8), the FOC with respect to variable C_t is simply:

$$f_C = J_X,$$

where f_C is the gradient of the aggregator (8) with respect to consumption. The approximate optimal portfolio rule is $C_t = \beta X_t$, not depending on the ambiguity parameter η.

$\square$

7.3.3 Optimal rules (27) and (28)

Considering the Bellman equation (26) and a value function with the form $J(X_t, y_t) = exp\left\{g\left(y_t, \sqrt{2\eta}\right)\right\}\frac{X_t^{1-\gamma}}{1-\gamma}$, the FOC with respect to π_t yields:

$$\pi_t = \frac{1}{\gamma}(\mu - r)y_t - \frac{\sqrt{1-\rho^2}\sqrt{2\eta}}{\gamma}\sqrt{y_t} + \frac{\sigma\rho}{\gamma}\frac{\partial g(y)}{\partial y}y_t. \qquad (44)$$

As for the general case in section 3, following perturbation theory, the function g is expanded in $\sqrt{2\eta}$ to first order. Expansion of function g is given in (18). Recalling from (18) that $g_0 = Ay_t + B$, with A and B given by (16) and (17), coincides with g when there is no ambiguity, and going back to (44), the asymptotic expansion for the optimal rule under ambiguity (28) is immediately obtained.

Regarding the optimal consumption rule (27), considering the Bellman (26) and the aggregator (8), the FOC with respect to variable C_t is again given by:

$$f_C = J_X,$$

where f_C is the gradient of the aggregator (8) with respect to consumption. The asymptotic optimal portfolio rule is therefore given by:

$$C_t = \beta X_t + \mathcal{O}_c^2\left(\sqrt{2\eta}\right),$$

which is (27).

$\square$

7.4 Bellman Equation for general contamination in section 4.2

From (29)-(30), for any admissible contamination $h = \begin{bmatrix} h^s & h^\varepsilon \end{bmatrix}^\top$, following the same steps as in section 2.3 the investment opportunity set and the intertemporal budget constraint faced by the ambiguous investor are deducted.

The corresponding Bellman equation is given by:

$$
\begin{aligned}
0 = \ & \sup_{\pi,C} \inf_{h^s,h^\varepsilon} \left\{ f\left(C,J\right) + \left[\pi_t \left(\mu + \sqrt{\frac{1}{y_t}} h^s - r \right) X_t + rX_t - C_t \right] J_X + \frac{1}{2}\pi_t^2 \frac{1}{y_t} X_t^2 J_{XX} + \right. \\
& \left. + \left[\kappa\left(\theta - y_t\right) + \sigma\sqrt{y_t}\rho h^s + \sigma\sqrt{y_t\left(1-\rho^2\right)} h^\varepsilon \right] J_y + \frac{1}{2}\sigma^2 y_t J_{yy} + \pi_t X_t \rho \sigma J_{Xy} \right\}.
\end{aligned}
\tag{45}
$$

Following the same approach as in appendix 7.1, Ξ represents the diffusion matrix of state variables y_t and X_t in their contaminated processes:

$$
\Xi = \begin{bmatrix} \sigma\sqrt{y_t}\rho & \sigma\sqrt{y_t\left(1-\rho^2\right)} \\[2ex] \pi_t\sqrt{\frac{1}{y_t}}X_t & 0 \end{bmatrix}.
\tag{46}
$$

The minimization of (45) with respect to the vector h gives:

$$
\begin{bmatrix} h^s \\[2ex] h^\varepsilon \end{bmatrix} = -\frac{\sqrt{2\eta}}{\sqrt{\begin{bmatrix} J_y & J_X \end{bmatrix} \Xi\Xi^\top \begin{bmatrix} J_y & J_X \end{bmatrix}^\top}} \Xi^\top \begin{bmatrix} J_y \\[2ex] J_X \end{bmatrix}.
\tag{47}
$$

Replacing (46) in (47), the vector of optimal contaminating drifts is obtained and given by:

$$
\begin{bmatrix} h^s \\ h^\varepsilon \end{bmatrix} = -\frac{\sqrt{2\eta}}{\sqrt{J_y^2 \sigma^2 y_t + 2 J_y J_X \sigma \rho \pi_t X_t + J_X^2 \pi_t^2 \frac{1}{y_t} X_t^2}} \begin{bmatrix} \sigma \sqrt{y_t} \rho J_y + \pi_t \sqrt{\frac{1}{y_t}} X_t J_X \\ \sigma \sqrt{y_t} \sqrt{1-\rho^2} J_y \end{bmatrix}.
$$

Substituting this result into (45), after some algebra, gives (14).

$\square$

References

Ahn, D., S. Choi, D. Gale, and S. Kariv: 2011, 'Estimating Ambiguity Aversion in a Portfolio Choice Experiment'. Unpublished manuscript.

Anderson, E. W., L. P. Hansen, and T. J. Sargent: 1998, 'Risk and Robustness in General Equilibrium'. Unpublished manuscript.

Arai, D.: 1997, 'Temporal resolution of uncertainty in risky choices'. *Acta Psychologica* **96**(1-2), 15–26.

Bansal, R. and A. Yaron: 2004, 'Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles'. *Journal of Finance* **59**(4), 1481–1509.

Blanchard, O.: 2009, '(Nearly) Nothing to Fear but Fear Itself'. *The Economist, January 31st.*

Bollerslev, T., R. Y. Chou, and K. F. Kroner: 1992, 'ARCH modeling in finance : A review of the theory and empirical evidence'. *Journal of Econometrics* **52**(1-2), 5–59.

Bossaerts, P., P. Ghirardato, S. Guarnaschelli, and W. R. Zame: 2010, 'Ambiguity in Asset Markets: Theory and Experiment'. *Review of Financial Studies* **23**(4), 1325–1359.

Buraschi, A., R. Kosowski, and F. Trojani: 2012, 'When There is No Place to Hide - Correlation Risk and the Cross-Section of Hedge Fund Returns'. Unpublished manuscript.

Buraschi, A., P. Porchia, and F. Trojani: 2010, 'Correlation Risk and Optimal Portfolio Choice'. *The Journal of Finance* **65**(1), 393–420.

Camerer, C. and M. Weber: 1992, 'Recent Developments in Modeling Preferences: Uncertainty and Ambiguity'. *Journal of Risk and Uncertainty* **5**(4), 325–70.

Campbell, J. Y. and L. M. Viceira: 2002, *Strategic Asset Allocation: Portfolio Choice for Long-Term Investors.* Oxford University Press.

Cao, H. H., T. Wang, and H. H. Zhang: 2005, 'Model Uncertainty, Limited Market Participation, and Asset Prices'. *Review of Financial Studies* **18**(4), 1219–1251.

Chacko, G. and L. M. Viceira: 2005, 'Dynamic Consumption and Portfolio Choice with Stochastic Volatility in Incomplete Markets'. *The Review of Financial Studies* **18**(4), 1369–1402.

Chen, H., N. Ju, and J. Miao: 2011, 'Dynamic Asset Allocation with Ambiguous Return Predictability'. Unpublished Manuscript.

Chen, Z. and L. Epstein: 2002, 'Ambiguity, Risk, and Asset Returns in Continuous Time'. *Econometrica* **70**(4), 1403–1443.

Chopra, V. K. . and W. T. Ziemba: 1993, 'The Effect of Errors in Means, Variances, and Covariances on Optimal Portfolio Choice'. *The Journal of Portfolio Management* **19**(2).

Cox, J. C., J. Ingersoll, Jonathan E, and S. A. Ross: 1985, 'A Theory of the Term Structure of Interest Rates'. *Econometrica* **53**(2), 385–407.

Duffie, D. and L. G. Epstein: 1992a, 'Asset Pricing with Stochastic Differential Utility'. *Review of Financial Studies* **5**(3), 411–36.

Duffie, D. and L. G. Epstein: 1992b, 'Stochastic Differential Utility'. *Econometrica* **60**(2), 353–94.

Ellsberg, D.: 1961, 'Risk, Ambiguity, and the Savage Axioms'. *The Quarterly Journal of Economics* **75**(4), 643–669.

Epstein, L. G. and M. Schneider: 2003, 'Recursive multiple-priors'. *Journal of Economic Theory* **113**(1), 1–31.

Epstein, L. G. and M. Schneider: 2010, 'Ambiguity and Asset Markets'. *Annual Review of Financial Economics* **2**, 315–346.

Epstein, L. G. and S. E. Zin: 1989, 'Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework'. *Econometrica* **57**(4), 937–69.

Epstein, L. G. and S. E. Zin: 1991, 'Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: An Empirical Analysis'. *The Journal of Political Economy* **99**(2), 263–286.

Faria, G. and J. Correia-da Silva: 2010, 'Robust Portfolio Choice with Ambiguity about Stochastic Volatility'. Unpublished manuscript.

Faria, G. and J. Correia-da Silva: 2012, 'The price of risk and ambiguity in an intertemporal general equilibrium model of asset prices'. *Annals of Finance* **8**(4), 507–531.

Faria, G., J. Correia-da Silva, and C. Ribeiro: 2009, 'Dynamic Consumption and Portfolio Choice with Ambiguity about Stochastic Volatility'. FEP Working Papers 348, Universidade do Porto, Faculdade de Economia do Porto.

Gagliardini, P., P. Porchia, and F. Trojani: 2009, 'Ambiguity Aversion and the Term Structure of Interest Rates'. *Review of Financial Studies* **22**(10), 4157–4188.

Garlappi, L., R. Uppal, and T. Wang: 2007, 'Portfolio Selection with Parameter and Model Uncertainty: A Multi-Prior Approach'. *Review of Financial Studies* **20**(1), 41–81.

Gollier, C.: 2011, 'Portfolio Choices and Asset Prices: The Comparative Statics of Ambiguity Aversion'. *Review of Economic Studies* **78**(4), 1329–1344.

Hansen, L. P. and T. J. Sargent: 1995, 'Discounted Linear Exponential Quadratic Gaussian Control'. *IEEE Trans. Automatic Control* (40), 968–971.

Hansen, L. P. and T. J. Sargent: 2001, 'Acknowledging Misspecification in Macroeconomic Theory'. *Review of Economic Dynamics* **4**(3), 519–535.

Hansen, L. P. and T. J. Sargent: 2006, 'Time Inconsistency of Robust Control?'. Unpublished manuscript.

Hansen, L. P., T. J. Sargent, G. Turmuhambetova, and N. Williams: 2006, 'Robust Control and Model Misspecification'. *Journal of Economic Theory* **128**(1), 45–90.

Hansen, L. P., T. J. Sargent, and N. E. Wang: 2002, 'Robust Permanent Income And Pricing With Filtering'. *Macroeconomic Dynamics* **6**(01), 40–84.

Kim, T. S. and E. Omberg: 1996, 'Dynamic Nonmyopic Portfolio Behavior'. *Review of Financial Studies* **9**(1), 141–61.

Knight, F. H.: 1921, *Risk, Uncertainty and Profit*. Houghton Mifflin, Boston.

Kogan, L. and R. Uppal: 2001, 'Risk Aversion and Optimal Portfolio Policies in Partial and General Equilibrium Economies'. Unpublished manuscript.

Kreps, D. M. and E. L. Porteus: 1978, 'Temporal Resolution of Uncertainty and Dynamic Choice Theory'. *Econometrica* **46**(1), 185–200.

Liu, J.: 2007, 'Portfolio Selection in Stochastic Environments'. *Review of Financial Studies* **20**(1), 1–39.

Maccheroni, F., M. Marinacci, and A. Rustichini: 2006, 'Ambiguity Aversion, Robustness, and the Variational Representation of Preferences'. *Econometrica* **74**(6), 1447–1498.

Maenhout, P. J.: 2004, 'Robust Portfolio Rules and Asset Pricing'. *Review of Financial Studies* **17**(4), 951–983.

Maenhout, P. J.: 2006, 'Robust Portfolio Rules and Detection-Error Probabilities for a Mean-Reverting Risk Premium'. *Journal of Economic Theory* **128**(1), 136–163.

Merton, R. C.: 1969, 'Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case'. *The Review of Economics and Statistics* **51**(3), 247–57.

Merton, R. C.: 1971, 'Optimum Consumption and Portfolio Rules in a Continuous-time Model'. *Journal of Economic Theory* (3), 373–413.

Merton, R. C.: 1973, 'An Intertemporal Capital Asset Pricing Model'. *Econometrica* **41**(5), 867–887.

Merton, R. C.: 1980, 'On Estimating the Expected Return on the Market: An Explanatory Investiga-
tion'. *Journal of Financial Economics* **8**, 323–361.

Savage, L.: 1954, *The Foundations of Statistics*. New York (2nd ed. 1972): John Wiley and Sons; New
York; Dover Publications.

Schroder, M. and C. Skiadas: 1999, 'Optimal Consumption and Portfolio Selection with Stochastic
Differential Utility'. *Journal of Economic Theory* **89**(1), 68–126.

Schroder, M. and C. Skiadas: 2003, 'Optimal Lifetime Consumption-Portfolio Strategies Under Trading
Constraints and Generalized Recursive Preferences'. *Stochastic Processes and their Applications*
108(2), 155–202.

Trojani, F. and P. Vanini: 2002, 'A Note on Robustness in Merton's Model of Intertemporal Consump-
tion and Portfolio Choice'. *Journal of Economic Dynamics and Control* **26**(3), 423–435.

Trojani, F. and P. Vanini: 2004, 'Robustness and Ambiguity Aversion in General Equilibrium'. *Review
of Finance* **8**(2), 279–324.

Ui, T.: 2011, 'The Ambiguity Premium vs. the Risk Premium under Limited Market Participation'.
Review of Finance **15**(2), 245–275.

Uppal, R. and T. Wang: 2003, 'Model Misspecification and Underdiversification'. *Journal of Finance*
58(6), 2465–2486.

von Neumann, J. and O. Morgenstern: 1944, *Theory of Games and Economic Behavior*. Princeton
University Press.

Xu, W., H. Li, and C. Wu: 2011, 'A Robust General Equilibrium Stochastic Volatility Model with
Recursive Preference Investors'. *Annals of Economics and Finance* (12-2), 217–231.