Violations In The Returns On European Options Under The Black Scholes Model

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Abstract

This paper describes the Itô processes for the continuously compounded returns on European call and put stock options under the one-dimensional diffusion assumption and the Black Scholes pricing model. It uses the Itô processes to motivate discrete time approximations for the returns on calls and puts. Theses models are used in a simulation study to compute the probability of an option return violation as defined by Bakshi *et* al (2000). Two specific cases are described in some detail. The main findings of the study are that, at both daily and intraday intervals, option returns are not perfectly correlated with underlying returns. Call (put) returns may move in the opposite (same) directions as that of the underlying return and call and put returns may move together. Even at high frequencies, such as the 30-minute sampling interval, some violation occurrence rates are not low, in particular for short-term and out of the money options. It may therefore be argued that the effect of time decay in short intervals is not always negligible and that the sign of the change in the price of an option may not be correctly predicted by the sign of the price change in the underlying stock. The findings confirm that violations are likely to present difficulties when using options for either hedging or speculating and that there is a need for further development of parametric models of option returns.

Key Words: Itô's lemma, Monte Carlo simulation, One-dimensional diffusions, Option returns, Violations.

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1. Introduction

The behaviour of the returns on options and the relationship to returns on the underlying stocks remains a relatively unexplored topic in the finance literature. As pointed out by Coval and Shumway (2001), relatively little work has focused on understanding the distribution of option returns. Sheikh and Ronn (1994), Coval and Shumway (2001) and Jones (2006) are among the few authors who have reported systematic research in understanding option returns. Other authors, Bates (1996) for example, have mentioned some properties of the entire return distribution of option positions briefly, but without further exploration. Broadie et al (2009) have computed the finite sample distribution of various option return statistics by using Monte Carlo simulation, and compared empirical option returns to those generated by the benchmark models. Goyal and Saretto (2009) have studied the cross-section of stock option returns by sorting stocks on the difference between historical realized volatility and implied volatility. Isakov and Morard (2001) have constructed portfolios of options and the underlying stocks, but have not studied the properties of option returns *per se*.

A significant exception to the relative lack of research on option returns is the important paper by Bakshi, Cao and Chen (2000), henceforth BCC. These authors note that all one-dimensional diffusion option-pricing theories imply that the underlying asset price is the sole source of uncertainty for the all of its options. Thus option prices must be perfectly correlated with each other and with the underlying asset; as the stock price rises or falls the price of a call (put) option moves in the same (opposite) direction. By contrast, when the underlying asset does not follow a one-dimensional diffusion, a call (put) premium can be a decreasing, concave (increasing, convex) function of the underlying asset price over some range. Thus call (put) premiums can move in the opposite (same) direction to the underlying asset price. BCC term such contrary movements violations. They describe a major empirical study into returns on the S&P500 index and call and put options written on the index and report a detailed analysis of violations. One of the conclusions is that option-pricing models based on one-dimensional diffusion are not the best way of modelling the returns on S&P500 index options. In a related work, Nordén (2001) studied daily data on Swedish (American) equity options data and, like BCC, found several violations of the expected properties. In their paper BCC also make the important point that the development of a parametric model for option returns is not only a logical step for researchers to take but is very desirable. They point out, for example, that an option violation can have the effect of making a hedged position more risky rather than less risky and that, by implication, the properties of option returns are of importance to traders and to fund managers. In other work on option returns, Coval and Shumway (2001) mainly check the relationship between options returns and strike prices. Broadie *et al* (2009) show that the expected option returns are determined by the underlying price and the options' elasticity which are functions primarily of moneyness and volatility.

This paper seeks to develop a better understanding of the return behaviour of European options from a theoretical perspective. The objective of this work is to provide evidence which will lead to the development of parametric models for option returns. This paper contributes to knowledge in the following ways. First, following Sheikh and Ronn (1994), Itô's lemma is used to motivate a simple model which provides an approximation for option returns subject to the usual assumptions of the Black-Scholes (1973, hereafter, BS) model. Secondly, this model is used to compute the theoretical probabilities of the option return violations as defined in BCC. Thirdly, the properties of the violations are studied using this model.

It is shown that violations do occur even under the standard IID normal assumption for stock returns (the one-dimensional diffusion) and the BS options pricing model. In the cases described in this paper, the incidence of violations depends on the characteristics of the contract. In some cases, the probability of a violation is indeed low, but in general the incidence of violations should not be neglected. The results reported in this paper extend BCC, Coval and Shumway (2001) and Broadie *et al* (2009) in a number of respects. In particular, it is shown that the probability of violation and the effect of time decay are both strongly time varying and increase considerably as expiry approaches. Furthermore, the effects are more substantial for out of the money options.

The structure of the paper is as follows. Section 2 presents the Itô processes for options prices and returns subject to the underlying assumptions of the BS model. In continuous time and subject to the assumptions made the Itô processes for option returns are exact. They motivate discrete time approximations. Section 3 summarises the definitions of the four types of violations which are given in BCC. The theoretical probabilities of violations are derived in Section 4 based on the model of Section 2. Section 5 describes the design parameters for a simulation experiment. Section 6 reports results for two specific cases. These have been chosen primarily to illustrate the effect of moneyness on option returns, since this appears to be a major determinant of violations. Section 7 describes the temporal behaviour of time decay and violation probability for these two cases. Section 8 concludes. In the interests of brevity and in keeping with increasingly common practice, only examples of key results are presented in the text of the paper. Further detail is available on request from the corresponding author. Technical assumptions and notation are that in common use.

2. Model For Option Returns Derived From Itô's Lemma

The price of the underlying stock time t is denoted by S_t. It is assumed that the stock pays no dividends over the life of the option, which matures at time $T \geq t$. The strike price is denoted by K. The return on the underlying asset between time t and $t+\Delta t$ is defined in the usual way as

$$R_S = ln(S_{t+\Delta t}) - ln(S_t). \quad (1.)$$

The price of a European call (put) at time t is denoted by C_t (P_t). the variables $d_{1,2}$ which are used below are defined in the usual way. The delta of the call (put) is denoted by Δ_C (Δ_P). The return on the call and put are defined as

$$R_C = ln(C_{t+\Delta t}) - ln(C_t), R_P = ln(P_{t+\Delta t}) - ln(P_t).$$

The price process for the BS model is

$$dS_t = \mu S_t dt + \sigma S_t dZ_t,$$

where μ is the drift, σ the return standard deviation and Z a Wiener process. The BS model also assumes risk neutrality, that is $\mu = r$, the risk free interest rate. Note that unless it is explicitly required the time subscript t is generally omitted from the notation. From Itô's lemma, the stock return follows the normal distribution

$$dlnS_t \sim N\{(r - \sigma^2/2)dt, \sigma^2 dt\}.$$

Applying Itô's lemma to C_t and P_t (as in Sheikh and Ronn, 1994) and using the well-known relationships between the gamma and theta of calls and puts gives the processes

$$dC_t = -rKe^{-r(T-t)}\Phi(d_2)dt + \Delta_C dS_t, \; dP_t = rKe^{-r(T-t)}\Phi(-d_2)dt + \Delta_P dS_t.$$

These show that the change in the call (put) price is positively (negatively) related to the change in the stock price and that there is negative (positive) drift. Applying Itô's lemma to lnC_t and lnP_t and simplifying gives the diffusions for the returns on calls and puts. These are

$$dlnC_t = \{r(1-\xi_C) + \sigma^2(\xi_C - \xi_C^2)/2\}dt + \xi_C dlnS_t,$$

$$dlnC_t = \{r(1-\xi_P) + \sigma^2(\xi_P - \xi_P^2)/2\}dt + \xi_P dlnS_t.$$

(2.)

where

$$\xi_C = \Delta_C S_t / C_t; \xi_P = \Delta_P S_t / P.$$

Equation (2.) indicates that option returns are combined functions of asset returns and time decay. In the case of calls, the time decay is less than or equal to zero. Numerically it is usually the case that the time decay for puts is less than zero. Mathematically, however, it is possible for the time decay in the puts component of (2.) to be positive. For instance, the time decay of an in-the-money European put option on a non-dividend-paying underlying asset may be positive or zero (Hull, 2006, p354). The processes defined at (2.) are exact and motivate models for the returns on call and put options in discrete time. Replacing dt by Δt and $dlnS_t$ by R_S as defined at (1.) gives the models

$$R_C = \{r(1-\xi_C) + \sigma^2(\xi_C - \xi_C^2)/2\}\Delta t + \xi_C R_s,$$

$$R_P = \{r(1-\xi_P) + \sigma^2(\xi_P - \xi_P^2)/2\}\Delta t + \xi_P R_S,$$

where R_C and R_P are respectively the return on the call and the put. For brevity, these are written as

$$R_C = \alpha + \beta R_S, R_P = \gamma + \delta R_S. \quad (3.)$$

where $R_S \sim N(\omega, \tau^2)$ with $\omega = (r-\sigma^2)\Delta t$ *and* $\tau^2 = \sigma^2 \Delta t$. Note that under the assumption of risk neutrality $\mu = r$. The coefficients in (3.) are given by

$$\alpha = \left\{r\left(1-\xi_C\right)+\sigma^2\left(\xi_C-\xi_C^2\right)/2\right\}\Delta t,\ \beta = \xi_C,$$

$$\gamma = \left\{r\left(1-\xi_P^2\right)+\sigma^2\left(\xi_P-\xi_P^2\right)/2\right\}\Delta t,\ \delta = \xi_P. \qquad (4.)$$

with $\beta \geq 1$ and $\delta \leq 0$.

3. Violations In Option Returns

In Equation (4.) α and γ are proportional to Δt. When the underlying asset price S_t follows the one-dimensional diffusion which leads to BS model, the following properties of European option returns are presented in BCC.

<u>Property 1:</u> Over a short time intervals in which α and γ are negligible, the return on an option is mostly in response to the contemporaneous return to the underlying asset. The return to the underlying asset and return on a European call written on it should share the same sign, $R_S R_C \geq 0$. For puts the signs should be opposite, $R_S R_P \leq 0$.

<u>Property 2:</u> Over any time intervals, the contemporaneous return on the underlying and time decay should be the only sources of uncertainty for option returns. That is, if $R_S = 0$, then $R_C \leq 0$ and $R_P \leq 0$ due to the time decay[1] effects.

<u>Property 3</u>: Over any short time intervals, contemporaneous returns on call and put options with the same strike price and the same maturity should be of opposite sign: $R_C R_P \leq 0$.

The above properties may be summarised as follows. After removing time decay, the returns on the underlying asset determine the returns on options written on it. The failure of price changes to comply with the above predictions of the model is termed a violation by BCC. As they report, the severity of violations is an important issue both for studies of option pricing and for the use of option return model in trading and portfolio construction. BCC further argue that, in the one-dimensional case, with sampling intervals ranging from 30 minutes to 1 day the time decay is negligible. They define four types of violations against this argument. These are as follows.

Type 1 Violation: For calls, $R_S R_C < 0$. There are the two cases of this type of violation, A: $R_S > 0$ but $R_C < 0$, B: $R_S < 0$ but $R_C > 0$. For puts, $R_S R_P > 0$ with cases A: $R_S > 0$ and $R_P > 0$, B: $R_S < 0$ and $R_P < 0$.

[1] Note that, as reported in Section 2, it is possible that the time decay for a put may be non-negative.

Type 2 Violation: For calls, $R_S R_C = 0$, $R_S \neq 0$. There are the two subcategories of this type of violation, A: $R_S > 0$, B: $R_S < 0$ For puts, $R_S R_P = 0$, $R_S \neq 0$ with the same sub-categories.

Type 3 Violation: For calls, $R_S R_C = 0$, $R_C \neq 0$. There are the two subcategories of this type of violation, A: $R_C > 0$, B: $R_C < 0$. For puts, $R_S R_P = 0$, $R_P \neq 0$ with the same sub-categories.

Type 4 Violation: $R_C R_P > 0$. There are the four subcategories of this type of violation, A: $R_S > 0$, $R_C > 0$ and $R_P > 0$, B: $R_S > 0$, $R_C < 0$ and $R_P < 0$, C: $R_S < 0$, $R_C > 0$ and $R_P > 0$, D: $R_S < 0$, $R_C < 0$ and $R_P < 0$.

The occurrence of Type 1 violations mean that option returns do not change monotonically with underlying return. Type 2 violations indicate that option returns do not change even after underlying return has changed. Type 3 violations directly show that contemporaneous underlying return and time decay are not the only sources of uncertainty for option returns. Type 4 violations mean that call returns and put returns move up or down together.

4. Probability Of Violations Using The Itô Based Model

The theoretical probabilities associated with each of the four types of violations in the BS model may be calculated using the models described at (3.) and (4.) in conjunction with the normal distribution function. The computations for the type 1 violations for calls are described in some detail. Other results are presented more briefly, but may be derived using similar methods. The notation *C1A* refers to a type 1 case A violation for a call. Related notation is defined in the same way.

Type 1 violations of call option returns

For a call option a type 1 violation occurs if

$$R_C R_S = (\alpha + \beta R_S) R_S < 0 .$$

For type 1 – case A the probability of a violation is

$$Pr(\mathrm{C}1A) = Pr\{(\alpha + \beta R_S) < 0, R_S > 0\} = Pr(0 < R_S < -\alpha/\beta).$$

Since $R_S \sim N(\omega, \tau^2)$ this is

$$Pr(\mathrm{C}1A) = \Phi\{(-\alpha/\beta - \omega)/\tau\} - \Phi\{-\omega/\tau\}.$$

The probability of a type 1 – case B violation is

$$Pr(\mathrm{C}1B) = Pr\{(\alpha + \beta R_S) > 0,\ R_S < 0\}.$$

Since α is non-positive the condition $\alpha + \beta R_S > 0$ implies that $R_S > -\alpha/\beta > 0$ which contradicts the condition $R_S < 0$. Hence

$$Pr(\mathrm{C}1B) = 0.$$

Type 1 violations of put option returns

For a put option a type 1 violation occurs if $R_P R_S > 0$. For cases A and B respectively the conditions are

$$(\gamma + \delta R_S) > 0, R_S > 0; (\gamma + \delta R_S) < 0, R_S < 0.$$

The probability of type 1 – case A and case B violations are respectively

$$Pr(\mathrm{P}1A) = Pr\{(\gamma + \delta R_S) > 0,\ R_S > 0\};\ Pr(\mathrm{P}1B) = Pr\{(\gamma + \delta R_S) < 0,\ R_S < 0\}.$$

There are three cases to consider; (1) $\gamma < 0$, (2) $\gamma = 0$ and (3) $\gamma > 0$. Case (3) could only occur for an in the money European put option for a non-dividend-paying underlying asset (Hull, 2006, p.354). For these cases, noting that $\delta < 0$ and proceeding as above gives the following results

Case (1) $\gamma < 0$

$$Pr(\mathrm{P}1\mathrm{A}) = 0;\ Pr(\mathrm{P}1B) = Pr(-\gamma/\delta < R_S < 0) = \Phi\{-\omega/\tau\} - \Phi\{(-\gamma/\delta - \omega)/\tau\},$$

Case (2) $\gamma = 0$

$$Pr(\mathrm{P}1\mathrm{A}) = 0;\ Pr(\mathrm{P}1\mathrm{B}) = 0,$$

Case (3) $\gamma > 0$

$$Pr(\mathrm{P}1A) = \Phi\{(-\gamma/\delta - \omega)/\tau\} - \Phi\{-\omega/\tau\};\ Pr(\mathrm{P}1B) = 0\,.$$

Type 2 and Type 3 violations of call and put option returns

A type 2 violation for calls (puts) occurs if $R_C = 0$ ($R_P = 0$) and a type 3 violation occurs if $R_S = 0$. Under the model of Section 2, R_S , R_C and R_P are continuous random variables and so these events occur with probability zero.

Type 4 violations of call/put option returns

Noting that $\alpha < 0$, $\beta \geq 1$, $\delta < 0$, there are also three cases for Type 4 violations.

Case (1) $\gamma < 0$

$$\Pr(4A) = Pr(R_S > 0, \alpha + \beta R_S > 0, \gamma + \delta R_S > 0) = Pr(R_S > -\alpha/\beta, R_S < -\gamma/\delta) = 0\,,$$

$$\Pr(4B) = Pr(R_S > 0, \alpha + \beta R_S < 0, \gamma + \delta R_S < 0) = \Phi\{(-\alpha/\beta - \omega)/\tau\} - \Phi\{-\omega/\tau\} = Pr(\mathrm{C}1A),$$

$$Pr(4C) = 0\,,$$

$$\Pr(4D) = Pr(R_S < 0, \alpha + \beta R_S < 0, \gamma + \delta R_S < 0) = \Phi\{-\omega/\tau\} - \Phi\{(-\gamma/\delta - \omega)/\tau\} = Pr(\mathrm{P}1B).$$

Case (2) $\gamma = 0$

$$Pr(4\mathrm{A}) = 0,\ Pr(4B) = \Phi\{(-\alpha/\beta - \omega)/\tau\} - \Phi\{-\omega/\tau\} = Pr(\mathrm{C}1A),\ Pr(4C) = 0,\ Pr(4\mathrm{D}) = 0.$$

Case (3) $\gamma > 0$

If $-\alpha/\beta < -\gamma/\delta$

$$\Pr(4A) = Pr(R_S > 0, \alpha + \beta R_S > 0, \gamma + \delta R_S > 0) = \Phi\{(-\gamma/\delta - \omega)/\tau\} - \Phi\{(-\alpha/\beta - \omega)/\tau\}$$

$$= \Pr(P1A) - P(C1A),$$

$$\Pr(4B) = 0\,.$$

If $-\alpha/\beta > -\gamma/\delta$, then

$$Pr(4A) = 0 ,$$

$$Pr(4B) = Pr(R_S > 0, \alpha + \beta R_S < 0, \gamma + \delta R_S < 0) = \Phi\{(-\alpha/\beta - \omega)/\tau\} - \Phi\{(-\gamma/\delta - \omega)/\tau\}$$

$$= Pr(C1A) - Pr(P1A).$$

and in both cases

$$Pr(4C) = Pr(4\text{D}) = 0 .$$

5. Simulation Experiment To Calculate Violation Probabilities

A simulation approach is used to calculate the violation probabilities specified in Section 4. The aim of the experiment is to create and use a simulated data set which is similar in structure to the data analysed by BCC. Table 1 lists the parameters employed in simulating the normal distribution and thus the BS model. Five different underlying expected return rates, risk free interest rates, values of annual volatility, and strike prices are employed.

Table 1 about here

As in BCC, five sampling frequencies are used: 30-minute, 1-hour, 2-hour, 3-hour, and 1 day. The simulated time period is 1 year, which contains 252 trading days. The number of trading hours per day is set to 8.5. For each simulation, the number of observations is inversely proportional to the sampling frequency; thus 252 for daily data rising to 4284 for the 30-minute sampling interval. The simulation and calculation procedures are as follows: (1) simulate the underlying return process; (2) convert the return series into asset price series; (3) calculate European call premiums with changing time to maturity using the BS formula; (4) use put-call parity to calculate European put premiums; (5) Compute the parameters α, β, γ and δ defined in equation (4.); (6) Compute the violation probabilities defined in Section 4.

The results reported in Section 6 below are based on 1,000 simulations for each parameter set and sampling frequency. The tables show results based on computing the average probability of a violation for each simulation; that is, the average over the time to maturity. According to the results in Section 4, type 2 and 3 violations occur with probability zero, as do some of the sub-categories of type 4 violations. In the following sections, therefore, only type 1 and the non-zero cases of the type 4 violations are reported.

6. Results Of The Simulation Experiments

This section contains tables which report selected results from the simulation experiments. The tables are constructed for the parameter sets in question by computing averages over the times to maturity for each simulation and then over the 1000 simulations. This means that the results in the tables may be compared with the corresponding tables in BCC. In addition to the average probabilities, the tables in this section also report basic statistics derived from the simulations. Two cases are presented below as exemplars of the substantial number of results. For both cases, the common parameter values are r = 0.1, σ = 0.3 and S_0 = 100. For case 1, μ = 0.15 and K = 110. For Case 2 μ = -0.15 and K_0 = 150. The values for r, μ and σ are annual values. The rationale for the values of the strike price K is discussed below after Table 2. As noted in the introduction, results for other parameter sets are available on request.

6.1 Violation occurrences across sampling frequencies

Table 2 reports the results of the simulations for Case 1 at the five sampling frequencies for the violation cases for which the probabilities computed using the formulae in Section 4 are non-zero.

Table 2 about here please

As Table 2 shows, sampling frequency is generally inversely related to violation rate. For a type 1A call violation and daily sampling frequency the average probability is 0.03 or 3%. At the 30-minute sampling frequency the average is about 0.9%. It may be noted that the standard deviation of the probability is also generally inversely related to sampling frequency. Thus, at 30 minutes the variability in violation probability is lower than that for daily sampling. In panel (i) of the table, the columns in Table 2 entitled *min* and *max* imply that the maximum type 1 case A violation probability for calls is about 10 times greater than the minimum. By contrast, in panel (ii) the maximum type 1 case B violation probability for puts is typically about 50 times greater than the minimum. It may also be noted that the results for calls in panel (i) and puts in panel (ii) are not the same at comparable sampling frequencies. Although it could be argued that the results in the *mean* column are similar, the standard deviations and maxima are clearly numerically different. In case 1, for which K = 110 and S_0 = 100, call options are initially out of the money. Since the mean rate of return $\mu = 0.15$, it is to be expected that many of the simulations will yield call prices which are in the money as expiry approaches. The opposite is likely to be the case for puts. The columns of Table 2 entitled *skew* and *kurt* indicate that the simulated distribution of violation probabilities for a put which is likely to be out of the money exhibit a greater degree of non-normality than that for the calls, which will tend to be in the money as expiry approaches.

Case 2 considers a more extreme situation. With K = 150 and μ = -0.15 call options are more likely than not to remain out of the money and puts in the money. The asymmetry in the results in the four panels of Table 3 therefore illustrates some of the effects of moneyness of calls and puts at the five sampling frequencies.

Table 3 about here please

In panels (ii) and (iii) of the table the mean violation probabilities for puts are small. Furthermore, the maximum values are small too. In short, for these in the money puts violations are rare. By contrast, the values in panels (i) and (iv) display higher violation probabilities which increase considerably as the sampling frequency decreases. As the *max* column of Table 3 shows, the maximum violation probabilities in panels (i) and (iv) are close to 50%. For out of the money calls, violations appear to be a more serious problem.

6.2 Violation occurrences across strike prices

To examine further the effect of strike price, Case 1 is repeated at the 30-minute sampling interval for strike prices 70, 90, 110,130 and 150. The other Case 1 parameters remain the same. The results are reported in Table 4.

Table 4 about here please

First note that the rows in both panels of Table 4 with strike price equal to 110 are the same as the 30-minute rows of the corresponding panels of Table 2. Panel (i) of Table 4 shows that the probability of Type 1A violations for calls (and Type 4B) is related to strike price. For in the money calls, the probability is low. As the strike price increases, so does the violation probability. Furthermore it becomes more volatile. As panel (i) shows for a deeply out of the money call with K = 150 the maximum violation probability is over 45%. Qualitatively, the opposite is true for puts although the numerical values do not exhibit any obvious symmetry. The columns of Table 4 entitled *skew* and *kurt* indicate that the distribution of simulated violation probabilities for out of the money calls and puts both exhibit substantial non-normality.

7. Violations Occurrences Across Time To Maturity

The averages reported in Section 6 conceal the effect of the passage of time on the probability of violations. Investigation of the theta of a European option shows the effect of diminishing time to expiry on the price and hence on return. In the models at (3.) and (4.) it is straightforward to show that, as the time to expiry tends to zero, the magnitudes of β and δ are unbounded if the option is out of the money. This section contains six figures which illustrate the effects of time decay. Each figure shows a time series for one simulation at the 30-minute sampling frequency. There are four figures for Case 1 and two figures for Case 2. As discussed above, in Case 1 the call is generally in the money or near the money and the put is generally out of the money or near the money.

Insert Figures 1 and 2 about here please

Figure 1 shows a time series of computed values of α (from equation 4). As the graph shows, time decay as measured by α is negative but very small in magnitude until a short time before expiry. After this it falls rapidly from a value close to -½% to –3%. Figure 2 shows a graph of the probability of a type 1 violation for calls for the same simulation. This probability increases from about ½% when the time to expiry is one year to a value of 4% or higher as expiry approaches. Visually, the probability in this simulation grows quadratically and its volatility increases as expiry approaches.

Insert Figures 3 and 4 about here please

Figure 3 shows a graph of γ (also from equation 4.) for the same simulation. The value remains very close to zero until expiry is close, after which its falls to a value of less than 10% is precipitate. Figure 4 shows the corresponding type 1B violation probability for the put in the same simulation. Similar to γ, the probability remains low, around ½% until expiry is close, after which it rises rapidly to a value of 9%. Figures 3 and 4 imply that the averages shown in the tables in Section 6 may contain valuable information about the general characteristics of violations, but that for trading activities the more detailed temporal data shown in the figures is of greater importance.

Insert Figures 5 and 6 about here please

Figures 5 and 6 are for Case 2.They show time series which are equivalent to those in Figures 1 and 2. For Case 2, in which the call is likely to be out of the money, α is close to zero until expiry is close. That is, in this simulation at least, the behaviour of α as shown in Figure 5 is qualitatively similar to that of γ in Figure 3. However, as Figure 5 shows, the decline of α is more severe. Figure 6 shows the probability of a type 1 violation for a call for Case 2. At one year to expiry, the probability is close to zero. However, it rises in an exponential manner until it is 50% or more as expiry approaches. Recalling that for Case 2 call options are likely to expire out of the money or even deeply out of them money illustrates the possible impact of this violation on trading strategies. In case 2, the put option is likely to expire in the money or deeply in the money. In this case, the values of γ and the violation probability which are equivalent to those shown for Case 1 in Figures 3 and 4 are very close to zero and so are omitted.

8. Conclusions

This paper describes the Itô processes for the continuously compounded returns on European call and put stock options under the one-dimensional diffusion assumption and the Black Scholes pricing model. It uses the Itô processes to motivate discrete time approximations for the returns on calls and puts. Theses models are used to compute the probability of an option return violation. As defined by Bakshi *et* al (2000) a violation describes a combination of option and stock returns which is contrary to that implied by the one-dimensional diffusion assumption. This paper then reports exemplars of the results of an extensive simulation study in which the incidence of violations, under the one-dimensional diffusion and Black Scholes pricing model assumptions, are investigated. Two specific cases are described in some detail. Other cases using different simulation parameters are omitted from the paper but provide results which are consistent with these two cases.

The main findings of the simulation study are that at both daily and intraday intervals, option returns are not perfectly correlated with underlying returns. Call (put) returns may move in the opposite (same) directions as that of the underlying return and call and put returns may move together. Even at high frequencies, such as the 30-minute sampling interval, some violation occurrence rates are not low, in particular for short-term and out of the money options. It may therefore be argued that the effect of time decay in short intervals is not always negligible and that the sign of the change in the price of an option may not be correctly predicted by the sign of the price change in the underlying stock. These findings extend the work of Bakshi *et al* (2000) in a number of ways. In particular, they confirm that violations are likely to present difficulties when using options for either hedging or speculating and that there is a need for further development of parametric models of option returns.

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Table 1 Parameters for Underlying Process and Option Prices in the BS Model

Parameter		Value
Annual expected asset returns	μ	-0.15, 0, 0.15, 0.3, 0.45
Annual Risk-free interest rate	r	0, 0.05,0.1,0.15,0.2
Annual volatility of asset returns	σ	0.1, 0.2, 0.3, 0.4, 0.5
Initial stock price	S_0	100
Changing time to maturity (year)	$T-t$	$253/252,\cdots,1/252$
Strike price	K	70, 90, 110, 130, 150

Table 2 Case 1 summary statistics for calculated violation occurrences at different sampling intervals

Case 1: $\mu = 0.15$, $r = 0.1$, $\sigma = 0.3$ and $S_0 = 100$, $K_0 = 110$, 1000 simulations

	1	**2**	**3**	**4**	**5**	**6**
Sampling interval	**mean**	**std**	**skew**	**kurt**	**min**	**max**
(i) Type 1 – A of call returns and Type 4 – B						
30-minutes	0.0094	0.0062	1.6798	5.0266	0.0040	0.0408
1-hour	0.0144	0.0110	1.6515	5.3888	0.0055	0.0625
2-hours	0.0132	0.0105	4.5052	28.5120	0.0071	0.0979
3-hours	0.0146	0.0078	3.8992	23.1992	0.0095	0.0909
1-day	0.0344	0.0130	1.9245	11.7020	0.0190	0.1231
(ii) Type 1 – B of put returns and Type 4 – D						
30-minutes	0.0047	0.0068	7.2068	64.1492	0.0016	0.0880
1-hour	0.0060	0.0079	8.5117	87.5160	0.0017	0.1040
2-hours	0.0153	0.0128	3.0786	15.8440	0.0031	0.1185
3-hours	0.0212	0.0230	4.0787	26.7622	0.0036	0.2116
1-day	0.0224	0.0331	3.7385	18.6873	0.0027	0.2297

The violation probabilities are computed using the formulae in Section 4. For each simulation at each sampling frequency an average probability (AP) is computed by averaging over the number of observations. For the 1-day sampling interval this is 252. For the 30-minute interval it is 252*17 = 4284. Column 1, mean, reports the average over the 1000 simulations of the AP values. Columns 2 through 6 report the standard basic statistics which are computed in the usual way. .

Table 3 Case 2 summary statistics for calculated violation occurrences at different sampling intervals

Case 2: μ = -0.15 , r = 0.1, σ = 0.3 and S_0 = 100, K_0 = 150, 1000 simulations

	1	2	3	4	5	6
Sampling interval	**mean**	**std**	**skew**	**kurt**	**min**	**max**
(i) Type 1 – A of call returns						
30-minutes	0.0330	0.0592	4.6917	29.1654	0.0064	0.4959
1-hour	0.0457	0.0730	3.7963	19.3621	0.0091	0.4944
2-hours	0.0502	0.0798	3.6728	17.6748	0.0131	0.4921
3-hours	0.0569	0.0850	3.4507	15.1840	0.0166	0.4903
1-day	0.0998	0.0997	2.3281	8.1775	0.0282	0.4837
(ii) Type 1 – A of put returns						
30-minutes	0.0010	0.0008	0.2829	1.7543	0.0000	0.0025
1-hour	0.0015	0.0014	0.3440	1.5854	0.0000	0.0040
2-hours	0.0007	0.0011	1.4620	3.5922	0.0000	0.0037
3-hours	0.0006	0.0010	2.0948	6.6012	0.0000	0.0042
1-day	0.0045	0.0027	-0.3572	1.8870	0.0000	0.0095
(iii) Type 1 – B of put returns and Type 4 - D						
30-minutes	0.0001	0.0002	2.3727	7.4427	0.0000	0.0009
1-hour	0.0002	0.0004	1.4627	3.6953	0.0000	0.0017
2-hours	0.0010	0.0010	0.5474	1.9459	0.0000	0.0041
3-hours	0.0008	0.0009	0.7464	2.3887	0.0000	0.0032
1-day	0.0000	0.0001	5.6279	34.9535	0.0000	0.0011
(iv) Type 4 – B						
30-minutes	0.0320	0.0589	4.7441	29.6375	0.0064	0.4943
1-hour	0.0442	0.0724	3.8640	19.8450	0.0091	0.4921
2-hours	0.0496	0.0790	3.7073	17.9188	0.0131	0.4887
3-hours	0.0564	0.0842	3.4647	15.3011	0.0166	0.4869
1-day	0.0953	0.0988	2.3774	8.3558	0.0282	0.4777

The values in this table are computed in the same way as those in Table 2.

Table 4 Case 1 Summary statistics of calculated violation occurrences at different strike prices

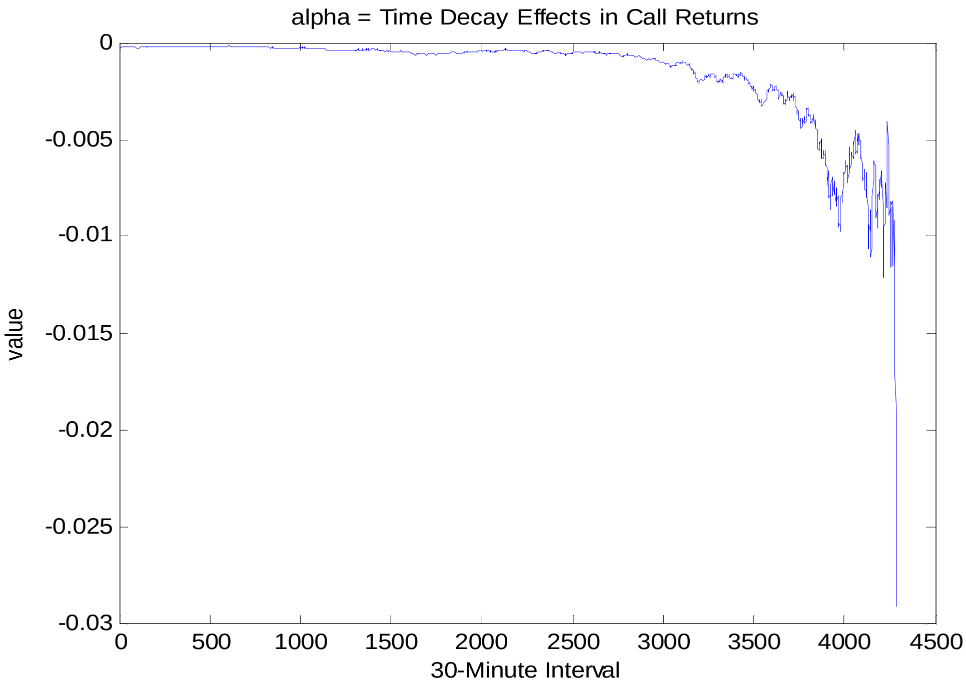
Case 1: $\mu = 0.15$, $r = 0.1$, $\sigma = 0.3$ and $S_0 = 100$, 30, 30-minute sampling interval, 1000 simulations

	1	2	3	4	5	6
Strike price	**mean**	**std**	**skew**	**kurt**	**min**	**max**
(i) Type 1 – A of call returns and Type 4 – B						
70	0.0029	0.0005	0.4400	2.3921	0.0020	0.0042
90	0.0051	0.0017	1.1279	3.6257	0.0029	0.0115
110	0.0094	0.0062	1.6798	5.0266	0.0040	0.0408
130	0.0163	0.0230	6.0167	54.8768	0.0051	0.3285
150	0.0227	0.0382	5.9612	49.3984	0.0061	0.4657
(ii) Type 1 – B of put returns and Type 4 - D						
70	0.0153	0.0173	3.8222	21.4893	0.0000	0.2035
90	0.0112	0.0212	6.6926	56.3425	0.0000	0.2776
110	0.0047	0.0068	7.2068	64.1492	0.0016	0.0880
130	0.0015	0.0007	1.4593	7.4646	0.0002	0.0058
150	0.0003	0.0003	0.6016	2.0132	0.0000	0.0013

The values in this table are computed in a similar was to those in Table 2.

Figure 1 - Alpha Time Decay Effect in Call Returns at 30-minute sampling interval for Case 1

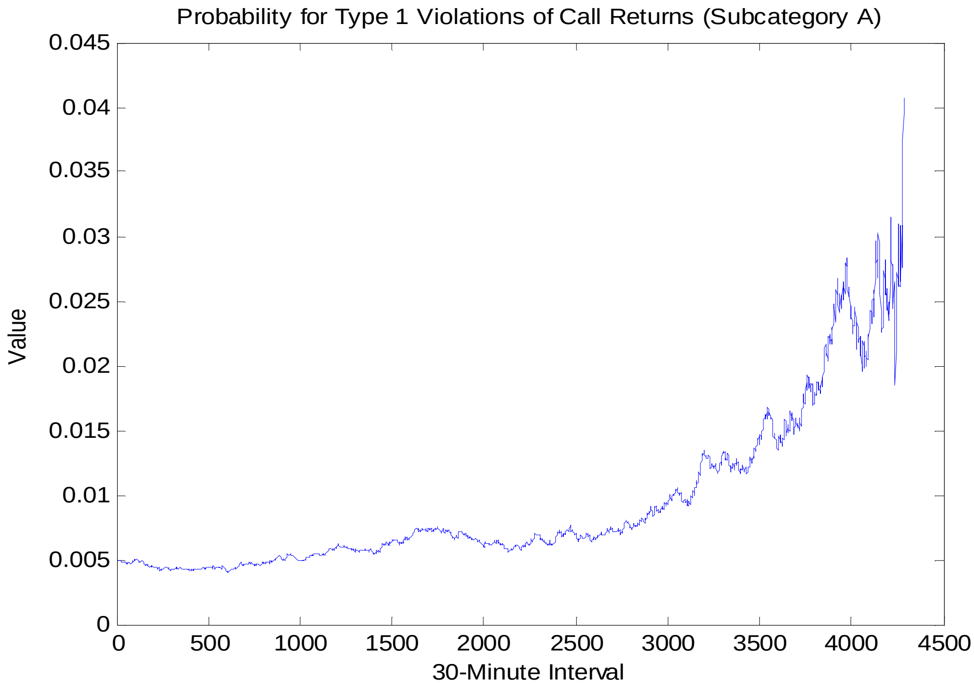
Case 1: $\mu = 0.15$, $r = 0.1$, $\sigma = 0.3$ and $S_0 = 100$, $K_0 = 110$, 1 simulation



The graph is the value of α as defined in equation (4.) for 1 simulation.

Figure 2 – Probability of a Type 1 Violation at 30-minute sampling interval for Call Returns for Case 1

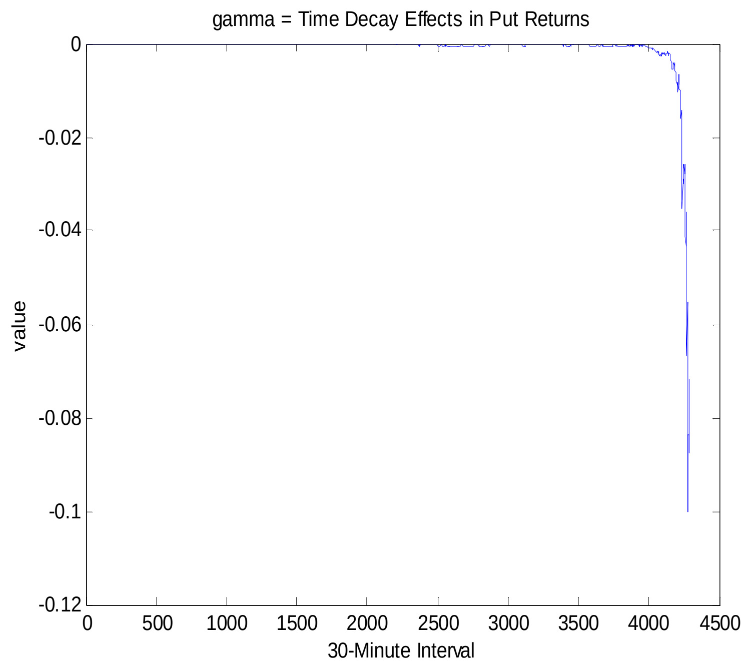
Case 1: $\mu = 0.15$, $r = 0.1$, $\sigma = 0.3$ and $S_0 = 100$, $K_0 = 110$, 1 simulations



The probability displayed is for one simulation and is computed using the methods of Section 4.

Figure 3 – Gamma Time Decay Effect in Put Returns at 30-minute sampling interval for Case 1

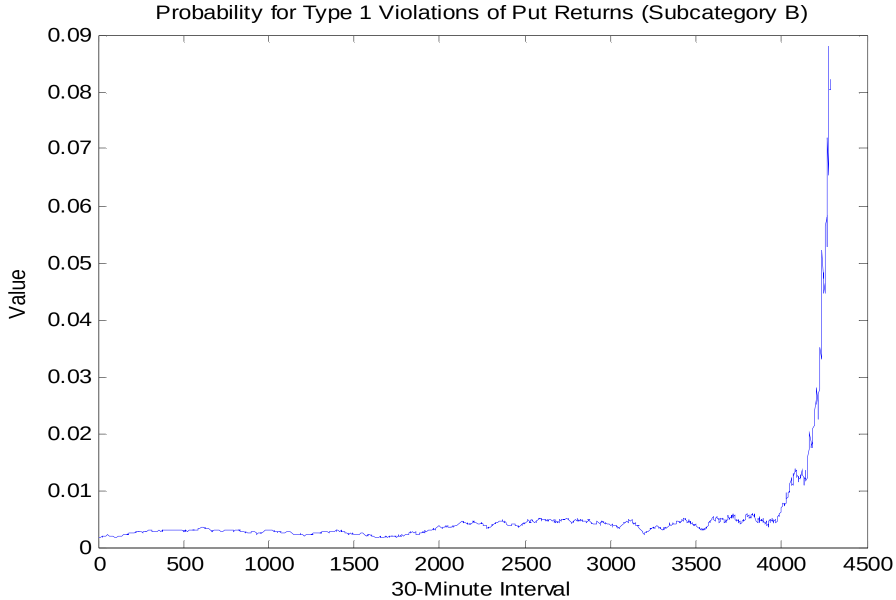
Case 1: $\mu = 0.15$, $r = 0.1$, σ = 0.3 and $S_0 = 100$, $K_0 = 110$, 1 simulations



The graph is the value of γ as defined in equation (4.) for 1 simulation.

Figure 4 – Probability of a Type 1B Violation at 30-minute sampling interval for Put Returns for Case 1

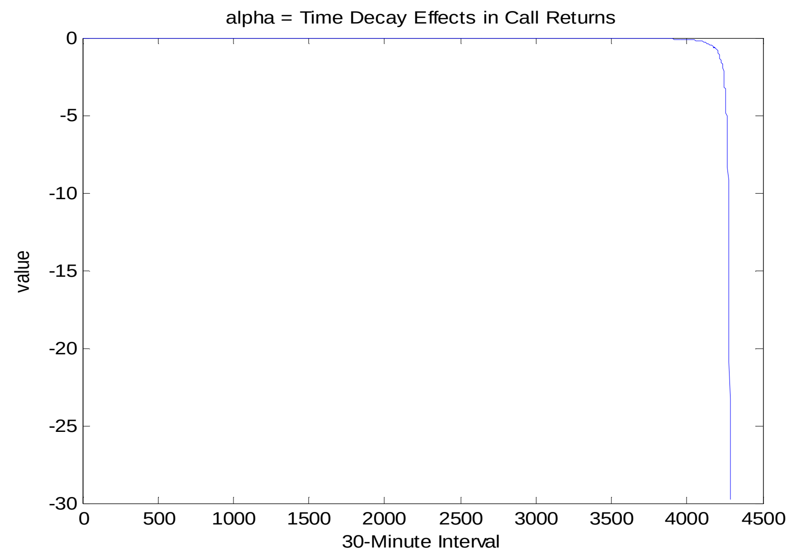
Case 1: $\mu = 0.15$, $r = 0.1$, $\sigma = 0.3$ and $S_0 = 100$, $K_0 = 110$, 1 simulations



The probability displayed is for one simulation and is computed using the methods of Section 4.

Figure 5 - Alpha Time Decay Effect in Call Returns at 30-minute sampling interval for Case 2

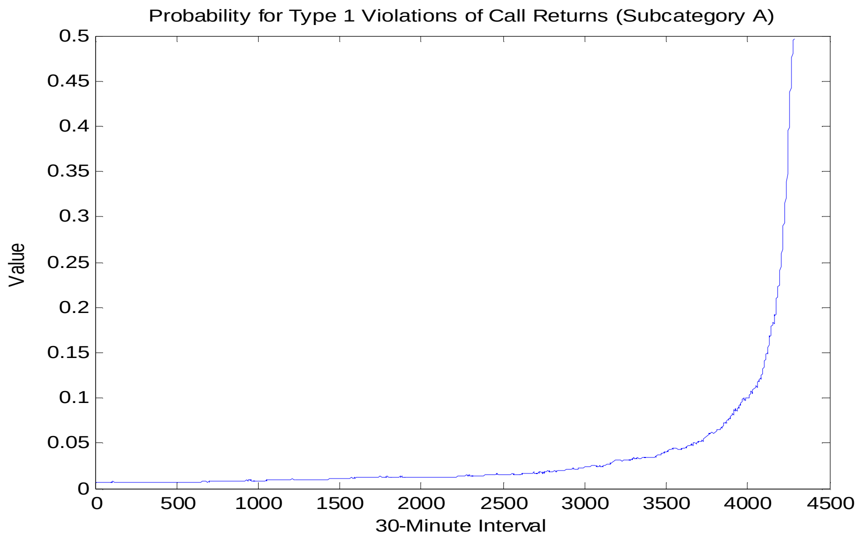
Case 2: $\mu = -0.15$, $r = 0.1$, $\sigma = 0.3$ and $S_0 = 100$, $K_0 = 150$, 1 simulation



The graph is the value of α as defined in equation (4.) for 1 simulation.

Figure 6 – Probability of a Type 1 Violation at 30-minute sampling interval for Call Returns for Case 2

Case 2: μ = -0.15 , r = 0.1, σ = 0.3 and S_0 = 100, K_0 = 150, 1 simulation



The probability displayed is for one simulation and is computed using the methods of Section 4.