

# Monetary Policy and Sovereign Debt Vulnerability

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## Abstract

We investigate quantitatively the effects of alternative monetary policy regimes on the sustainability of nominal sovereign debt, using a continuous-time open economy model. In our setup, default happens if the debt ratio surpasses an optimally determined default threshold. Under our baseline calibration we find that, at relatively low debt ratios, an anti-inflationary commitment is preferable to creating inflation so as to deflate debt away. However, for debt ratios sufficiently close to default, it is optimal to abandon the anti-inflationary commitment, as this permits debt stabilization and avoids a collapse in bond prices. Both strategies are in turn dominated by a mixed strategy: announcing that the anti-inflationary commitment will be kept unless the debt ratio surpasses an optimally determined threshold.

*Keywords:* optimal stopping, monetary-fiscal interactions, default.

*JEL codes:* E5, E62,F34.

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# 1 Introduction

One of the main legacies of the 2007-9 financial crisis and the subsequent recession has been the emergence of large fiscal deficits across the industrialized world. The consequence has been a sharp increase in government debt, with debt-to-GDP ratios near or above record levels in countries such as the United States, United Kingdom, Japan or the Euro area periphery (Greece, Spain, Italy, Ireland and Portugal). Before the summer of 2012, Euro area periphery economies experienced dramatic spikes in their sovereign yields, whereas other highly indebted countries did not. Many observers emphasized that a key difference between both groups of countries was that, whereas the US, UK and Japan had the option to deflate away the real burden of nominal debt through inflation, the Euro countries were forced to repay debt solely through fiscal surpluses.

These developments raise the question as to what role monetary policy should have, if any, in guaranteeing the sustainability of sovereign debt. Broadly speaking, on the one hand it can be argued that central banks should provide a ‘monetary backstop’ that reassures investors in sovereign debt.<sup>1</sup> This would bring sovereign yields down to more sustainable levels, thus giving governments embarked on fiscal consolidation strategies the possibility of implementing them more gradually. On the other hand, such a course of action may presumably give rise to inflation, with the resulting associated costs and distortions.

In this paper, we try to shed light on this question by studying quantitatively the trade-offs between price stability and sovereign debt sustainability. With this purpose, we build a general equilibrium continuous-time model of a small open economy in which a benevolent government issues long-term sovereign bonds to foreign investors. The government may default on sovereign debt if it finds it optimal to do so.<sup>2</sup> Default produces some costs due to permanent exclusion from capital markets and a drop in the output endowment. We assume *nominal* debt and we let the government choose the inflation rate. Thus, the government can deflate debt away. However, inflation also entails a direct welfare cost and raises nominal bond yields through the pricing decisions of investors.

Working in continuous time is quite convenient in order to solve the model. The government’s problem is one of *optimal stopping* as the decision to default is a discrete one. In addition, the government chooses inflation, a continuous variable. We show that the optimal default decision is characterized by a *default threshold* for the model’s single state variable, the debt-to-GDP ratio: an increase in the latter above such threshold triggers default. The equilibrium is characterized by a coupled system of two ordinary differential equations: a Hamilton-Jacobi-Bellman equation for the government and a bond pricing equation for the investors. We introduce a new numerical

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<sup>1</sup>This view is shared e.g. by Krugman (2011) or De Grauwe (2011).

<sup>2</sup>Fiscal policy also involves a fiscal rule for the primary deficit, where the latter responds to deviations in the debt ratio from a long-run target.

algorithm to solve this problem, based on recent work on finite difference methods by Achdou et al. (2014).<sup>3</sup>

After calibrating the model to an average industrialized economy, we first investigate the consequences of two baseline monetary strategies: one that commits to keeping inflation at zero at all times and thus renounces to deflate debt away (*anti-inflationary* regime), and one that determines inflation optimally in a discretionary manner (*no-commitment* regime). We find that, for relatively low debt levels, the anti-inflationary commitment achieves superior welfare outcomes, as it avoids inflationary distortions and an inflation premium in bond yields, while keeping default risks at a minimum. However, as debt approaches the default threshold under the anti-inflationary regime and investors start perceiving default as rather imminent, bond prices fall rapidly and not before long default materializes. At such debt levels, the no-commitment policy performs better than the anti-inflationary one: by using inflation to stabilize the debt ratio, it makes default much less likely and thus avoids a collapse in bond prices, the benefits of which outweigh the implied inflation costs. In fact, we show that the no-commitment policy *raises* the optimal default threshold, thus making sovereign debt more sustainable at relatively high debt ratios.

We then study the effects of a third monetary strategy that combines features of the two baseline policies just discussed: the government announces a commitment to keeping inflation at zero *unless* the debt ratio surpasses a certain threshold, at which point it switches to the no-commitment policy. This strategy, which can be interpreted as a conditional or loose commitment, is inspired by the ECB's Outright Monetary Transactions (OMT) programme, which was announced in July of 2012, at the height of the European debt crisis.<sup>4</sup> We thus refer to such a strategy as *OMT*. Importantly, the debt threshold above which the government switches to the no-commitment policy, or *OMT threshold*, is determined optimally, analogously to the default threshold. In order to jointly solve for both thresholds, we extend the optimal-stopping numerical techniques discussed above.

We find that the OMT strategy dominates the two baseline regimes for any debt ratio, by optimally combining their respective virtues. At relatively low debt ratios, for which the switch to the inflationary policy has not taken place yet, the OMT announcement avoids the costs of inflation while keeping default risks to a minimum. For debt ratios close to the OMT threshold, investors' anticipation of an imminent abandonment of the anti-inflationary commitment manages to stave off default fears, thus avoiding a collapse in bond prices. This latter effect dominates the costs from the inflation hike that takes place at the OMT threshold. Above the OMT threshold, the government simply follows the no-commitment strategy, which as explained before allows for

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<sup>3</sup>Our numerical algorithm guarantees that the value function and bond prices converge to the viscosity solution, which is the appropriate concept of a generalized solution in stochastic control problems.

<sup>4</sup>The ECB's announcement was broadly interpreted as implying a (non-explicit) threshold for bond yields or spreads, as opposed to debt ratios, for the purpose of activating OMTs. In our model, however, the existence of a monotonically decreasing equilibrium relationship between debt ratios and bond prices implies that a yield- or spread-based threshold is equivalent to a debt-based one.

a higher default threshold and thus a more sustainable sovereign debt.

To summarize our baseline results, an anti-inflationary commitment is clearly desirable at ‘manageable’ debt ratios, by keeping inflationary distortions to a minimum while posing no challenge for debt sustainability. However, if debt comes sufficiently close to the default threshold associated to the anti-inflationary commitment, it pays off to abandon such a commitment and start using inflation so as to stabilize the debt ratio. And it is even more desirable to *announce* that such a switch in monetary policy will take place if the debt ratio rises sufficiently, regardless of how low current debt levels are, due to the effect of this announcement on investors’ expectations.

Finally, we investigate the robustness of our baseline results to variations in some of the key model parameters, such as the toughness of the fiscal rule, the average maturity of government bonds, and the welfare costs of inflation. We find that our results are broadly robust, although with some qualifications. For very short bond maturities, both baseline monetary regimes imply very similar outcomes. Intuitively, since short maturities imply that large fractions of the debt stock must be renovated within short intervals of time, the increase in nominal bond yields produced by an inflationary policy affects also a larger share of the debt stock. This makes it too costly to create inflation so as to stabilize debt, with optimal inflation falling to zero even at moderate debt ratios. Likewise, if the response of primary deficit to debt deviations from target is very weak, both monetary strategies imply nearly identical default thresholds. The reason is that, if the government is sufficiently ‘irresponsible’ in its fiscal rule, beyond a certain level the debt ratio starts spiralling out of control, at which point it is not worthwhile to use inflation so as to avoid default.

**Literature review.** The possibility of employing inflation as an alternative to sovereign default has been present in most European economies since, at least, the Renaissance. This is well documented by Kindleberger (1984) and Reinhart and Rogoff (2009). Before the widespread use of paper currency, it was typically done through currency debasement. The first experience with debt deflation through inflation in a fiat-money economy was that of the John Law’s System in the 1720s France, as discussed by Velde (2007). Since the early 1900s the number of cases of high inflation in the face of fiscal problems has been considerable. A famous example is the German hyperinflation of the 1920s and its connections to the war reparations (see, for instance, Ferguson and Granville, 2000). In the face of these experiences, we provide a coherent theory of the trade-offs between debt deflation and sovereign default.

Our paper also relates to recent theoretical papers that analyze the link between sovereign debt vulnerability and monetary policy or inflation credibility, such as Aguiar, Amador, Farhi and Gopinath (2013, 2014) or Corsetti and Dedola (2014). In contrast to these papers, which consider self-fulfilling debt crises along the lines of Calvo (1988) or Cole and Kehoe (2000), in our model sovereign default is an optimal decision made by the government, as in Eaton and

Gersowitz (1981).<sup>5</sup> Also, while those papers are theoretical, we adopt a quantitative approach. In both respects, our model is more in line with the contributions by Aguiar and Gopinath (2006) or Arellano (2008). As mentioned before, however, while the latter consider real economies, we introduce instead nominal bonds and let the government choose the inflation rate, which allows us to study the effects of alternative monetary strategies on sovereign debt sustainability.<sup>6</sup>

Finally, this paper contributes to the literature on stochastic control models with fixed costs. Stokey (2009) provides an excellent introduction to such models and many useful references. We extend this literature in two fronts. First, by analyzing a coupled system in which the government's decision to default or deflate affects investors' bond pricing, which in turn modifies the debt dynamics. Second, by providing new numerical routines to solve this mathematical structure.

The structure of the paper is as follows. In section 2 we introduce the model. In section 3 we solve the two baseline monetary scenarios. In section 3 we introduce the OMT strategy. In section 4 we provide some robustness checks. Section 5 concludes.

## 2 Model

We consider a small, open, continuous-time, infinite-horizon economy.

### 2.1 Output, price level and sovereign debt

Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \hat{P})$  be a filtered probability space. There is a single, freely-traded consumption good which has an international price normalized to one. The economy is endowed with  $Y_t$  units of good each period (real GDP). The evolution of  $Y_t$  is given by

$$dY_t = \mu Y_t dt + \sigma Y_t dW_t, \tag{1}$$

where  $W_t$  is a  $\mathcal{F}_t$ -Brownian motion,  $\mu \in \mathbb{R}$  is the drift parameter and  $\sigma \in \mathbb{R}_+$  is the volatility. We assume that  $Y_0 = 1$ .

The local currency price relative to the World price at time  $t$  is denoted  $P_t$ . It evolves according to

$$dP_t = \pi_t P_t dt, \tag{2}$$

where  $\pi_t$  is the instantaneous inflation rate. We normalize the initial price level to one:  $P_0 = 1$ .

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<sup>5</sup>In Corsetti and Dedola (2014) default crisis can also be due to weak fundamentals.

<sup>6</sup>Another important difference is that our model is in continuous time, whereas those of Aguiar and Gopinath (2006) or Arellano (2008) are in discrete time. The use of continuous-time methods is quite convenient to analyze default decision as they are a particular case of optimal-stopping problems, which makes the computation quite efficient. In addition, by using continuous-time methods the state of the system can be summarized by a single state variable, the debt-to-gdp ratio.

The government trades a nominal non-contingent bond with risk-neutral competitive foreign investors. Let  $B_t$  denote the outstanding stock of nominal bonds (public debt). We assume that in every period, a fraction  $\lambda \in (0, 1)$  of the principal is paid back, while the remaining  $(1 - \lambda)$  is left outstanding. This means that the debt has an expected life of  $1/\lambda$ . The stock of debt thus evolves as follows,

$$dB_t = B_t^{new} dt - \lambda dt B_t,$$

where  $B_t^{new}$  is the flow of new debt issued at time  $t$ . The nominal market price of debt at time  $t$  is  $Q_t$ . Government debt pays a proportional coupon  $\delta$  per period. Also, the government incurs a nominal primary deficit  $P_t(C_t - Y_t)$ , where  $C_t$  is aggregate consumption.<sup>7</sup> The government's flow of funds constraint is then

$$Q_t B_t^{new} = (\lambda + \delta) B_t + P_t (C_t - Y_t).$$

That is, the proceeds from issuance of new bonds must cover amortization and coupon payments plus the primary deficit. Combining the last two equations, we obtain the following dynamics for nominal debt outstanding,

$$dB_t = \left[ \frac{B_t}{Q_t} (\lambda + \delta) - \lambda B_t + \frac{P_t}{Q_t} (C_t - Y_t) \right] dt. \quad (3)$$

We define the debt-to-GDP ratio as  $b_t \equiv B_t / (P_t Y_t)$ . Its dynamics are obtained by applying Itô's lemma to equations (1)-(3),

$$db_t = \left[ \left( \frac{\lambda + \delta}{Q_t} + \sigma^2 - \mu - \lambda - \pi_t \right) b_t + \frac{c_t}{Q_t} \right] dt - \sigma b_t dW_t, \quad (4)$$

where  $c_t \equiv (C_t - Y_t) / Y_t$  is the primary deficit-to-GDP ratio. Equation (4) describes the evolution of the debt-to-GDP ratio as a function of the primary deficit ratio, inflation and the bond price.

## 2.2 Government

The government has preferences over paths for aggregate consumption and domestic inflation given by

$$U_0 \equiv E_0 \left[ \int_0^\infty e^{-\rho t} u(C_t, \pi_t) dt \right]. \quad (5)$$

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<sup>7</sup>As in Arellano (2008), we assume that the government rebates back to households all the net proceedings from its international credit operations (i.e. its primary deficit) in a lump-sum fashion. Denoting by  $T_t$  the primary deficit, we thus have  $P_t C_t = P_t Y_t + T_t$ . This implies  $T_t = P_t (C_t - Y_t)$ .

For tractability, we assume that instantaneous utility takes the form

$$u(C_t, \pi_t) = \log(C_t) - \frac{\psi}{2}\pi_t^2, \quad (6)$$

where  $\psi > 0$ . Therefore, the government's welfare criterion penalizes deviations of inflation from a zero target. Using  $C_t = (1 + c_t)Y_t$ , we can express welfare in terms of the primary deficit ratio  $c_t$  as follows,

$$\begin{aligned} U_0 &= E_0 \left[ \int_0^\infty e^{-\rho t} \left( \log(1 + c_t) + \log(Y_t) - \frac{\psi}{2}\pi_t^2 \right) dt \right] \\ &= E_0 \left[ \int_0^\infty e^{-\rho t} \left( \log(1 + c_t) - \frac{\psi}{2}\pi_t^2 \right) dt \right] + V_0^{aut}, \end{aligned}$$

where

$$V_s^{aut} \equiv E_s \left[ \int_s^\infty e^{-\rho(t-s)} \log(Y_t) dt \right] = \frac{\log(Y_s)}{\rho} + \frac{\mu - \sigma^2/2}{\rho^2} \quad (7)$$

is the (exogenous) value of being in autarky at time  $s \geq 0$ .<sup>8</sup> Thus, the government's welfare criterion increases with the primary deficit ratio and decreases with inflation deviations from zero. We also assume that political constraints prevent the government from having a negative debt, that is,  $b_t \geq 0$ . This means that instead of accumulating net foreign claims the government decides to increase consumption.

**Fiscal policy.** In order to determine its deficit, the government follows a fiscal rule. The primary deficit over GDP is set according to

$$c_t = \phi (\bar{b} - b_t), \quad (8)$$

where  $\bar{b} > 0$  is a target level for the debt-to-GDP ratio and  $\phi > 0$  is the rate of adjustment. This fiscal rule forces the government to run a primary surplus if  $b_t > \bar{b}$  and vice versa.

**Optimal default.** At any moment the government may decide to default on its debt. In case of default, the government is punished by permanent loss of access to the international capital markets, and the country suffers a permanent contraction in its output endowment. From then onwards, the government's flow of funds constraint is simply  $C_t = (1 - \kappa)Y_t$ , where  $\kappa$  is the percentage output loss following default. Since there is no debt to deflate, optimal inflation following

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<sup>8</sup>Notice that (1) and Itô's Lemma imply  $d \log Y_t = (\mu - \sigma^2/2) dt + \sigma dW_t$ . Solving for  $\log Y_t$  and taking time- $s$  conditional expectations yields  $E_s(\log Y_t) = \log Y_s + (\mu - \sigma^2/2)(t - s)$ , which combined with the definition of  $V_s^{aut}$  gives us the right-hand side of (7).

default is zero,  $\pi_t = 0$ . Therefore, the value of defaulting at a given time  $s \geq 0$  is given by

$$U_s^{def} = E_s \left[ \int_s^\infty e^{-\rho(t-s)} (\log(1 - \kappa) + \log(Y_t)) dt \right] = V + V_s^{aut},$$

where  $V \equiv \log(1 - \kappa) / \rho$  is the value of defaulting net of the autarky value.

Denote by  $T$  the time to default. Every period the government decides whether to default or not and chooses the path for inflation  $\pi$  in order to maximize its utility. This is an *optimal stopping* problem, as one of the policy instruments is a stopping time.<sup>9</sup> The solution to this kind of problem is characterized by an “inaction region” of the state variable,  $\Phi \equiv [0, b^*)$ , in which the government chooses not to default, and a region  $[b^*, \infty)$  in which the government defaults. The value function (net of the autarky value  $V_0^{aut}$ ) given an initial debt ratio  $b$  is given by

$$V(b) = \max_{\pi, b^*} E_0 \left[ \int_0^{T(b^*)} e^{-\rho t} \left( \log(1 + c_t) - \frac{\psi}{2} \pi_t^2 \right) dt + e^{-\rho T(b^*)} V | b_0 = b \right], \quad (9)$$

subject to debt dynamics (4).  $T(b^*)$ , the time to default, is a stopping time with respect to the filtration  $\{\mathcal{F}_t\}$  defined as the smallest time  $T$  such that  $b_T = b^* : T = \min(t, b_t = b^*)$ . The optimal default threshold  $b^*$  must satisfy the following two conditions,

$$V(b^*) = V, \quad (10)$$

$$V'(b^*) = 0. \quad (11)$$

Equation (10) is the *value matching* condition and it requires that, at the default threshold  $b^*$ , the value of continuing to honor the debt repayments equals the value of defaulting. Equation (11) is the *smooth pasting* (or smooth fit) condition, and it requires that there is no kink at the optimal default threshold. These conditions imply that the value function is continuous and continuously differentiable:  $V \in C^1([0, \infty))$ .<sup>10</sup>

The solution of this problem is given by the Hamilton-Jacobi-Bellman (HJB) equation,

$$\rho V(b) = \max_{\pi} \log(1 + c) - \frac{\psi}{2} \pi^2 + \left[ \left( \frac{\lambda + \delta}{Q(b)} + \sigma^2 - \mu - \lambda - \pi \right) b + \frac{c}{Q(b)} \right] V'(b) + \frac{(\sigma b)^2}{2} V''(b), \quad (12)$$

$\forall b \in [0, b^*)$ , and the boundary conditions (10) and (11).<sup>11</sup>

**Monetary policy.** We consider two alternative monetary policy regimes:

1. *No commitment.* In this case, the government chooses the inflation rate at each point in

<sup>9</sup>See e.g. Dixit and Pindyck (1994) or Stokey (2009).

<sup>10</sup>In addition to these two boundary conditions, there is the state constraint  $b \geq 0$  discussed above.

<sup>11</sup>Obviously,  $\forall b \in [b^*, \infty)$ ,  $V(b) = V$ .



time in order to maximize the value function in (12). That is, the government follows a *discretionary* optimal monetary policy. The optimal inflation rate is given by the first order condition:

$$\pi = -\frac{b}{\psi}V'(b). \quad (13)$$

2. *Anti-inflationary commitment.* In the second policy regime the government establishes an independent anti-inflationary central bank which commits to keeping inflation at zero at all times,  $\pi_t = 0$ . This is a particular case of (12) in which no optimization is taken with respect to  $\pi$ . The government may still decide whether or not to default on its debt, but it commits not to use inflation to deflate it away.

### 2.3 International investors

The government contracts with competitive risk-neutral international investors with the same discount factor  $\rho$  as the domestic government. In case of default, investors only manage to recover a share  $\theta < \frac{\lambda+\delta}{\rho+\lambda}$  of nominal debt outstanding. The unit price of the nominal non-contingent bond described above is

$$Q(b) = E_0 \left[ \int_0^{T(b^*)} e^{-(\rho+\lambda)t - \int_0^t \pi_s ds} (\lambda + \delta) dt + e^{-(\rho+\lambda)T(b^*) - \int_0^{T(b^*)} \pi_s ds} \theta \mid b_0 = b \right], \quad (14)$$

where again  $T$  denotes the smallest time to default. Combining this expression with Itô's lemma, we obtain

$$Q(b) (\rho + \pi(b) + \lambda) = (\lambda + \delta) + \left[ \left( \frac{\lambda + \delta}{Q(b)} + \sigma^2 - \mu - \lambda - \pi \right) b + \frac{c(b)}{Q(b)} \right] Q'(b) + \frac{(\sigma b)^2}{2} Q''(b), \quad (15)$$

for all  $b \in [0, b^*)$ , with boundary condition

$$Q(b^*) = \theta. \quad (16)$$

The partial differential equation (15), together with the boundary condition (16), provide the risk-neutral pricing of nominal defaultable bonds.<sup>12</sup>

Given a current bond price  $Q_t$ , the implicit bond yield  $r_t$  is the discount rate of a riskless real bond with the same price. Since the price of a riskless real bond with discount rate  $r_t$  is

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<sup>12</sup>There is also the state constraint  $b \geq 0$ .

$\int_t^\infty e^{-(r_t+\lambda)(\tau-t)} (\lambda + \delta) d\tau = \frac{\lambda+\delta}{r_t+\lambda}$ ,  $r_t$  is the solution to

$$\frac{\lambda + \delta}{r_t + \lambda} = Q_t \Leftrightarrow r_t = \frac{\lambda + \delta}{Q_t} - \lambda. \quad (17)$$

The *spread* with respect to the riskless rate is then  $spr_t \equiv r_t - \rho = \frac{\lambda+\delta}{Q_t} - \lambda - \rho$ . We can also calculate *total* nominal government deficit,

$$D_t \equiv P_t (C_t - Y_t) + \delta B_t + (1 - Q_t) \lambda B_t,$$

i.e. the sum of nominal primary deficit, coupon payments ( $\delta B_t$ ), and the price discount ( $(1 - Q_t)$ ) on the new bonds issued to pay for maturing bonds ( $\lambda B_t$ ). The total deficit-to-GDP ratio then equals

$$d_t \equiv \frac{D_t}{P_t Y_t} \equiv c_t + (\lambda + \delta) b_t - Q_t \lambda b_t = c_t + r_t Q_t b_t,$$

where in the last equality we have used the yield definition in (17).

## 2.4 Equilibrium

We define our equilibrium concept:

**Definition 1** *A Competitive Equilibrium is an interval  $\Phi = [0, b^*)$ , a value function  $V(b) : \Phi \rightarrow \mathbb{R}$ , an inflation policy function  $\pi : \Phi \rightarrow \mathbb{R}$  and a bond price  $Q : \Phi \rightarrow \mathbb{R}_+$  such that*

1. *Given prices  $Q$ , for any initial debt  $b_0 \in \Phi$  the value function  $V$  solves the government problem (12), the optimal inflation is  $\pi$  and the optimal debt threshold is  $b^*$ .*
2. *Given the optimal inflation  $\pi$  and the interval  $\Phi$ , bond prices satisfy the pricing equation (15).*

The government takes the bond price as given and chooses inflation (continuous policy) and default (stopping policy) to maximize its value function. The investors take these policies as given and price the new bond issues accordingly. Up to our knowledge, we are the first ones to study such a mathematical structure in continuous time.

## 3 The impact of alternative monetary policy regimes

### 3.1 Calibration

We calibrate the model in order to loosely replicate the dynamics of an average advanced economy. The unit of time is the year. We set the intertemporal discount factor  $\rho$  (i.e. the riskless real interest

rate) to a standard value of 4 percent. Based on international evidence in Moro (2014), the drift and diffusion parameters of the GDP process are set to  $\mu = 0.025$  and  $\sigma = 0.035$ .<sup>13</sup>

The bond parameters  $(\delta, \lambda, \theta)$  are set as follows. The recovery rate after default,  $\theta$ , is set to 30 percent, close to the value targeted by Yue (2010).<sup>14</sup> The coupon is set equal to the risk-free rate,  $\delta = \rho$ , such that the price of the riskless real bond  $\frac{\delta+\lambda}{\rho+\lambda}$  is normalized to 1. Finally, the amortization rate  $\lambda$  is set such that the average bond duration for a riskless bond, defined as

$$\frac{\int_0^\infty t e^{-(\rho+\lambda)t} (\lambda + \delta) dt}{\int_0^\infty e^{-(\rho+\lambda)t} (\lambda + \delta) dt} = \frac{1}{\rho + \lambda},$$

is 10 years. This yields  $\lambda = 0.06$ . Below we consider the effects of alternative durations.

The fiscal rule is calibrated in the spirit of the European Stability and Growth pact. The debt ratio target  $\bar{b}$  is set to 0.6 so that for debt levels above 60 percent of GDP the government is forced to run a primary surplus and vice versa. To calibrate the response coefficient  $\phi$ , we target a half-life of 12 years for a deviation in the debt ratio from its target.<sup>15</sup> This gives us  $\phi = 0.074$ . However, given the lack of explicit rules on the celerity with which debt deviations must be corrected, we later perform sensitivity analysis with respect to  $\phi$ .

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<sup>13</sup>Moro (2014) reports an average trend growth of 2.6% across upper-middle income countries and 2.3% across high income countries. Likewise, he finds average standard deviations in GDP growth of 3.8% and 3.1%, respectively.

<sup>14</sup>Yue (2010) reports an average recovery rate of 27% in the 2005 Argentinian sovereign default.

<sup>15</sup>We define the half-life of a debt deviation as the time it takes for such deviation to shrink by half in expectation. In the special case of riskless real bonds ( $Q_t = 1, \pi_t = 0$ ), it can be shown that  $E(b_t - b_{ss} | b_0) = e^{\mu_b t} (b_0 - b_{ss})$ , where  $\mu_b \equiv \delta + \sigma^2 - \mu - \phi < 0$  and  $b_{ss} \equiv \lim_{t \rightarrow \infty} E(b_t | b_0) = \phi \bar{b} / (-\mu_b)$  are respectively the drift and the long-run level of the debt ratio. Thus the half-life  $t^*$  solves

$$\frac{e^{\mu_b t^*} (b_0 - b_{ss})}{b_0 - b_{ss}} = \frac{1}{2} \Leftrightarrow t^* = \frac{\log(2)}{\phi - \delta - \sigma^2 + \mu}.$$

Table 1. Baseline calibration

Parameter	Value	Description
$\rho$	0.04	discount rate
$\mu$	0.025	drift output growth
$\sigma$	0.035	diffusion output growth
$\lambda$	0.06	bond amortization rate
$\delta$	0.04	bond coupon rate
$\theta$	0.30	bond recovery rate
$\bar{b}$	0.60	fiscal rule, debt/GDP target
$\phi$	0.074	fiscal rule, adjustment coefficient
$\psi$	40	inflation disutility parameter
$\kappa$	0.06	output loss from default

We set the inflation disutility parameter  $\psi$  to 40. This value, together with the other baseline parameter values, implies that a government with a debt ratio of 100 percent keeps inflation slightly below 2 percent, which seems plausible. In section 5 we investigate the consequences of choosing alternative values for  $\psi$ .

Finally, we set the percentage output loss from default  $\kappa$  to 0.06, which falls within the range of estimates in the literature.<sup>16</sup> The implied value of default is  $\bar{V} = -1.55$ . Table 1 summarizes the calibration.

### 3.2 Computational algorithm

The model proposed above does not yield an analytical solution. Therefore we resort to numerical techniques in order to find a solution. The optimal stopping nature of the problem guarantees that  $V \in C^1([0, \infty))$  but not to  $C^2$ . The appropriate concept of a general solution for stochastic optimal control is that of a *viscosity solution* (Crandall and Lions, 1983; Crandall, Ishii and Lions, 1992). The mathematical structure of the model is of particular interest as it comprises a pair of coupled ordinary difference equations (ODEs): the HJB equation (12) and the bond pricing equation (15). Here we propose a computational algorithm aimed at finding the equilibrium. It requires the numerical solution of the HJB and the bond pricing equations.

In order to solve the HJB and bond pricing equations, we employ an upwind finite difference method. It approximates the value function  $V(b)$  and the bond price  $Q(b)$  on a finite grid with steps  $\Delta b$ :  $b \in \{b_1, \dots, b_I\}$ , where  $b_i = b_{i-1} + \Delta b, \forall i \geq 2$ . We use the notation  $V_i \equiv V(b_i)$ ,  $Q_i \equiv Q(b_i)$ ,

<sup>16</sup>Aguiar and Gopinath (2006) and Yue (2010) assume a 2% output drop following default. Mendoza & Yue (2012) target an output loss from default of 14%.

$i = 1, \dots, I$ . Barles and Souganidis (1991) have proved how the proposed finite difference methods converge to the unique viscosity solution of the problem.

In order to compute the numerical solution to the recursive competitive equilibrium we proceed in three steps. We consider an initial guess of the bond price  $Q^0$  and the threshold  $b^{*0}$ . Set  $n = 1$ . Then:

1. **Government problem.** Given  $Q^{n-1}$  and  $b^{*n-1}$ , we solve the optimal stopping problem with a variable control. This means solving the HJB equation (12) in the domain  $[0, b^{*n-1}]$  imposing the smooth pasting condition (11) (but not the value matching condition) to obtain an estimate of the value function  $V^n$  and of the inflation  $\pi^n$ .
2. **Investors problem.** Given  $\pi^n$  and  $b^{*n-1}$ , solve the bond pricing equation (15) and obtain  $Q^n$  in the domain  $[0, b^{*n-1}]$ . Then iterate again 1 and 2 until both the bond price and the value function converge for a given  $b^{*n-1}$ .
3. **Optimal boundary.** Given  $V^n$  from step 2, compute  $V^n(b^{*n-1})$ . If  $V^n(b^{*n-1}) > \underline{V}$ , then increase  $b^{*n}$  and set  $n := n + 1$ . If  $V^n(b^{*n-1}) < \underline{V}$ , then decrease  $b^{*n}$ . Proceed again to steps 1 and 2 until the value matching condition  $V(b^*) = \underline{V}$  is satisfied.

Appendix A provides further details about steps 1 and 2. The idea of the algorithm is to find the equilibrium numerically by moving the boundary (the default threshold) and solving the HJB and bond pricing equations. The algorithm stops when the value function at the boundary coincides with  $\underline{V}$ .

### 3.3 Results

We solve the model for the two monetary regimes: anti-inflationary, and no commitment. Results are displayed in Figure 1. The upper left panel displays the value function. For debt-to-GDP ratios below 160 percent, the anti-inflationary regime (red line) yields a slightly higher value than the no-commitment regime (dashed green line). However, for higher debt ratios the no commitment regime significantly outperforms the anti-inflationary one. In fact, the value of sticking to zero inflation falls very rapidly, hence it also reaches the default value  $\underline{V}$  very quickly. Conversely, under no commitment the possibility of using inflation to deflate debt away allows for a gradual (essentially linear) decline in the value function. This has an important implication: the default threshold  $b^*$  for the debt ratio is higher for the no commitment case (around 240 percent; see green circle in the plot) than for the anti-inflationary one (around 180 percent; red circle). In other words, while generating worse outcomes at ‘normal’ debt levels, the no commitment monetary regime makes debt more sustainable once it reaches relatively high levels.

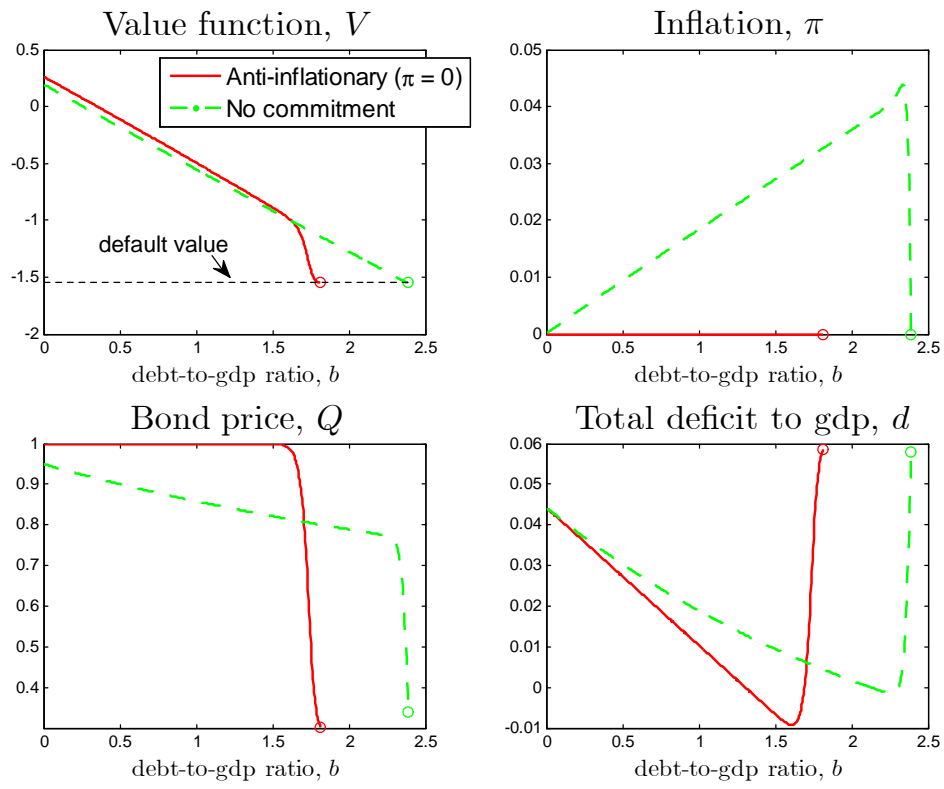


Figure 1: Equilibrium for different monetary policy regimes

The upper right panel of Figure 1 displays the optimal inflation rate. In the anti-inflationary regime it is always zero. In the no-commitment regime, it has an inverse U-shape, being zero at both boundaries ( $b = 0$  and  $b = b^*$ ) and reaching a maximum of 4.4 percent near the default threshold.<sup>17</sup> Intuitively, the incentive to deflate the debt away increases almost linearly with the debt ratio. However, for debt ratios close enough to default, the inflation required to stabilize debt is simply too costly in terms of welfare losses, and thus the government opts for abandoning debt deflation and focusing only on inflation stabilization.

The bond price is displayed in the lower left panel of Figure 1.<sup>18</sup> For debt-to-GDP ratios below 170 percent, bond prices in the no commitment regime are lower than in the zero inflation one, or equivalently bond yields are higher, which mostly captures the compensation for expected inflation required by investors. However, for higher debt ratios the situation reverses, with bond prices falling very quickly in the anti-inflationary regime as sovereign default is perceived as rather imminent by investors. Bond prices have an immediate reflection on the total deficit to GDP ratio,  $d_t$  (lower right panel). For relatively low debt levels, the government incurs a lower deficit under the anti-inflation commitment, thanks to higher bond prices.<sup>19</sup> But as debt reaches its critical level, plummeting bond prices drive deficit to unsustainable levels.

Figure 2 displays the drift of the debt-to-GDP ratio, given by  $s(b) \equiv \left( \frac{\lambda + \delta}{Q(b)} + \sigma^2 - \mu - \lambda - \pi(b) \right) b + \frac{c(b)}{Q(b)}$ , with  $c(b) = \phi(b - \bar{b})$ . The drift determines the dynamics of  $b_t$  in the absence of output shocks. In each monetary regime, the drift function crosses the zero line once from above and once from below. The first crossing point constitutes a stable equilibrium, the second one an unstable one. Notice that the stable equilibrium is very similar in both regimes: 74 percent under no commitment, and 77 percent under the zero inflation commitment. However, dynamics are very different for debt ratios above 174 percent: while debt rises under zero inflation (positive drift), it falls under the discretionary optimal policy (negative drift) except for values close enough to default.

Figure 3 displays the expected time to default  $\tau(b) \equiv E[T|b_t = b]$  given a current debt ratio  $b$ . Its computation is described in Appendix B. In both monetary regimes, for initial debt ratios below 150 percent the expected time to default is in the order of thousands of years. That is, sovereign debt is virtually safe. For debt ratios above 150 percent, however, expected time to default falls very rapidly under the anti-inflationary commitment, reaching zero (by construction)

<sup>17</sup>Notice that the optimal inflation level is given by the first order condition  $\pi = -\frac{b}{\psi}V'(b)$ . The maximum of inflation is  $b_\pi : \pi' = -\frac{1}{\psi}V'(b_\pi) - \frac{b_\pi}{\psi}V''(b_\pi) = 0 \Rightarrow b_\pi = -\frac{V'(b_\pi)}{V''(b_\pi)}$ .

<sup>18</sup>The green circle indicating the bond price at the threshold is slightly above its theoretical value,  $Q(b^*) = \theta = 0.30$ . This is due only to the discrete approximation of the state space and the fact that thresholds do not fall exactly on any of the grid points. Increasing the number of grid points sufficiently would eventually allow the numerical threshold bond price to coincide with its theoretical value. This also applies to other figures in the paper.

<sup>19</sup>Between about 130% and 169%, total deficit is actually negative under the anti-inflationary commitment. The reason is that the *primary surplus* that the government if forced to run under its fiscal rule,  $c_t = \phi(b_t - 0.60)$ , dominates the effect from decreasing bond prices.

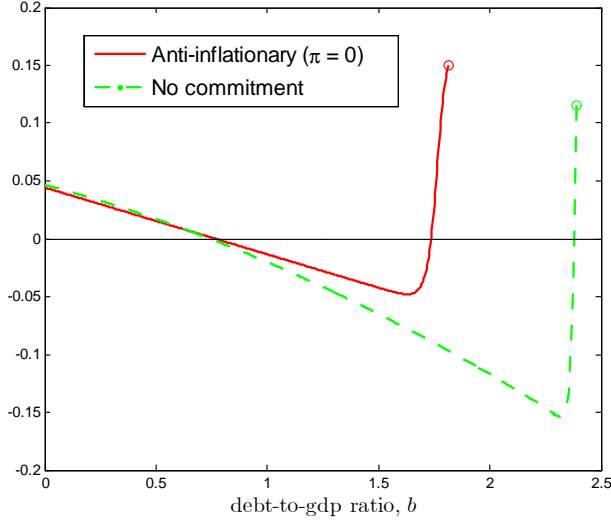


Figure 2: Drift of debt-to-GDP ratio,  $s(b)$

at the default threshold of 181 percent. Under the no-commitment policy, the time to default remains at very high levels, but also falls very quickly as debt approaches its default level.

So far we have analyzed the equilibrium of the endogenous variables as a function of the model's state variable, the debt-to-GDP ratio  $b$ . We now turn our attention to the model's dynamic behavior. With this purpose, we simulate the model using a discrete-time approximation of the law of motion of  $b_t$ , given by

$$b_{t+\Delta t} = b_t + \left[ \left( \frac{\lambda + \delta}{Q(b_t)} + \sigma^2 - \mu - \lambda - \pi(b_t) \right) b_t + \frac{c(b_t)}{Q(b_t)} \right] \Delta t - \sigma b_t \sqrt{\Delta t} \varepsilon_{t+\Delta t}, \quad (18)$$

where  $\varepsilon_t \sim N(0, 1)$ . We use a weekly frequency, such that  $\Delta t = 1/52$ . We use Monte Carlo simulation so as to sample the distribution of the endogenous variables at each point in time, conditional on an initial debt ratio  $b_0$ .<sup>20</sup> Figure 4 displays the *median* path of the debt ratio and bond price, together with the 95 percent confidence band, for two different initial debt ratios:  $b_0 = 1.50$  (left column) and  $b_0 = 1.75$  (right column).<sup>21</sup> Starting from a 150 percent debt ratio, the median path (black line) of the debt ratio declines gradually. The upper bound of the confidence interval remains also at a stable level before starting a very slow decline; in fact, we verify that the government does not default in any of our simulations (within the simulation horizon). Consequently, bond prices (lower left panel) stay always at or very close to one, the price of a riskless real bond.

<sup>20</sup>In particular, we simulate the model for 520 weeks (10 years) and repeat the simulation 2,000 times.

<sup>21</sup>Strictly speaking, upon default outstanding debt  $b_t$  falls to zero and the sovereign debt market closes down, such that the bond price  $Q_t$  in fact no longer exists. However, in Figures 4 and 5, for visual convenience we assume that upon default the debt ratio and the bond price both remain at their respective boundary values,  $b_t = b^*$  (where  $b^*$  is endogenous to the monetary regime) and  $Q_t = \theta = 0.30$ .



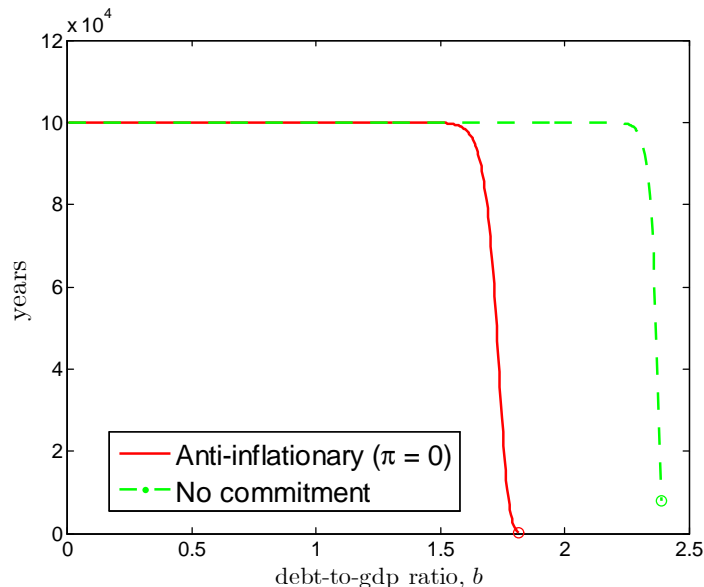


Figure 3: Time to default

Things are very different though if the debt ratio starts at 175 percent, i.e. close to the default threshold  $b^* = 1.81$ . As shown by the upper right panel, half of the times the government defaults before the 40<sup>th</sup> week. The imminence of default has its bearing on forward-looking bond prices, the median path of which starts collapsing very quickly after a few weeks.

Figure 5 displays the same dynamic quantiles under the no-commitment optimal monetary policy. The left column sets the initial debt ratio to  $b_0 = 1.75$ , which makes it directly comparable to the right column of Figure 4. Unlike in the case of the anti-inflationary commitment, under no commitment the economy does not default in any of our simulations (again, within the simulation horizon). This is due of course to the fact that the no-commitment policy raises substantially the default threshold (2.39) relative to the one in the anti-inflationary case (1.81). In the median case, the debt ratio declines gradually. As the debt burden declines, the benefits from deflating debt away become smaller and smaller compared to the welfare costs of inflation. As a result, the optimal median inflation rate (lower panel) also declines smoothly. Anticipating lower inflation in the future, investors in turn are willing to pay higher and higher prices for nominal government bonds (middle panel).

Even more striking are the results in the right column of Figure 5, where the initial debt ratio is set to  $b_0 = 2.30$ , i.e. very close to the default threshold  $b^* = 2.39$ . As shown by the upper panel, in 97.5 percent of the simulations the economy does *not* default within the simulation horizon.<sup>22</sup> As a consequence, while the bond price (middle panel) may occasionally experience temporary declines due to relatively high default risk, most of the times it eventually stabilizes at levels that

<sup>22</sup>We find that, within our simulation horizon of 5 years, default happens in only 2.2% of our simulations.

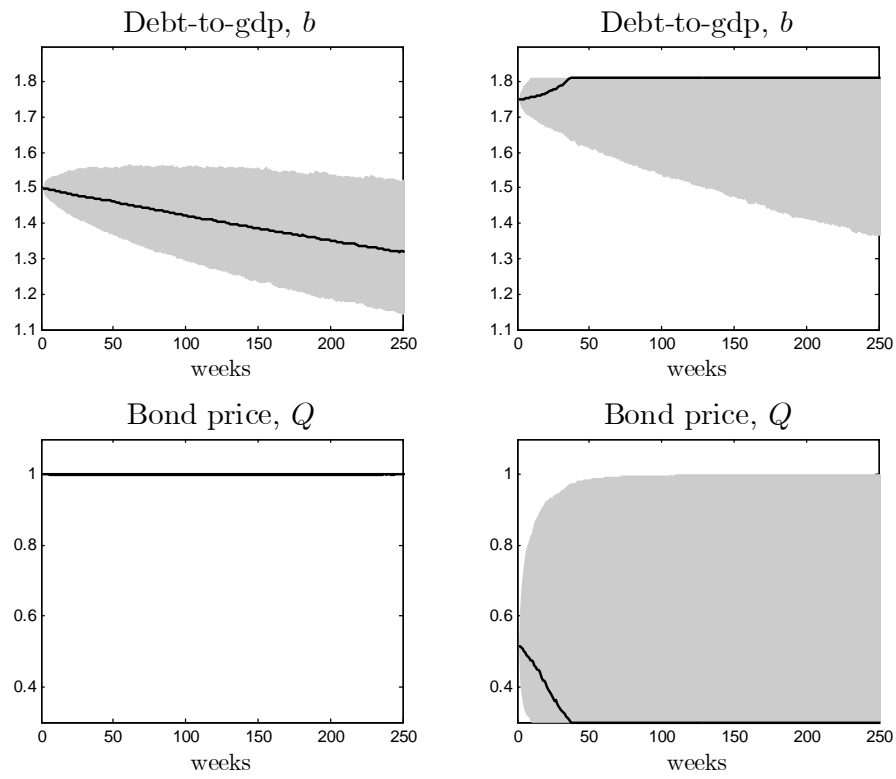


Figure 4: Dynamics under anti-inflationary policy. Initial state  $b_0 = 1.50$  (left column) and  $b_0 = 1.75$  (right column)

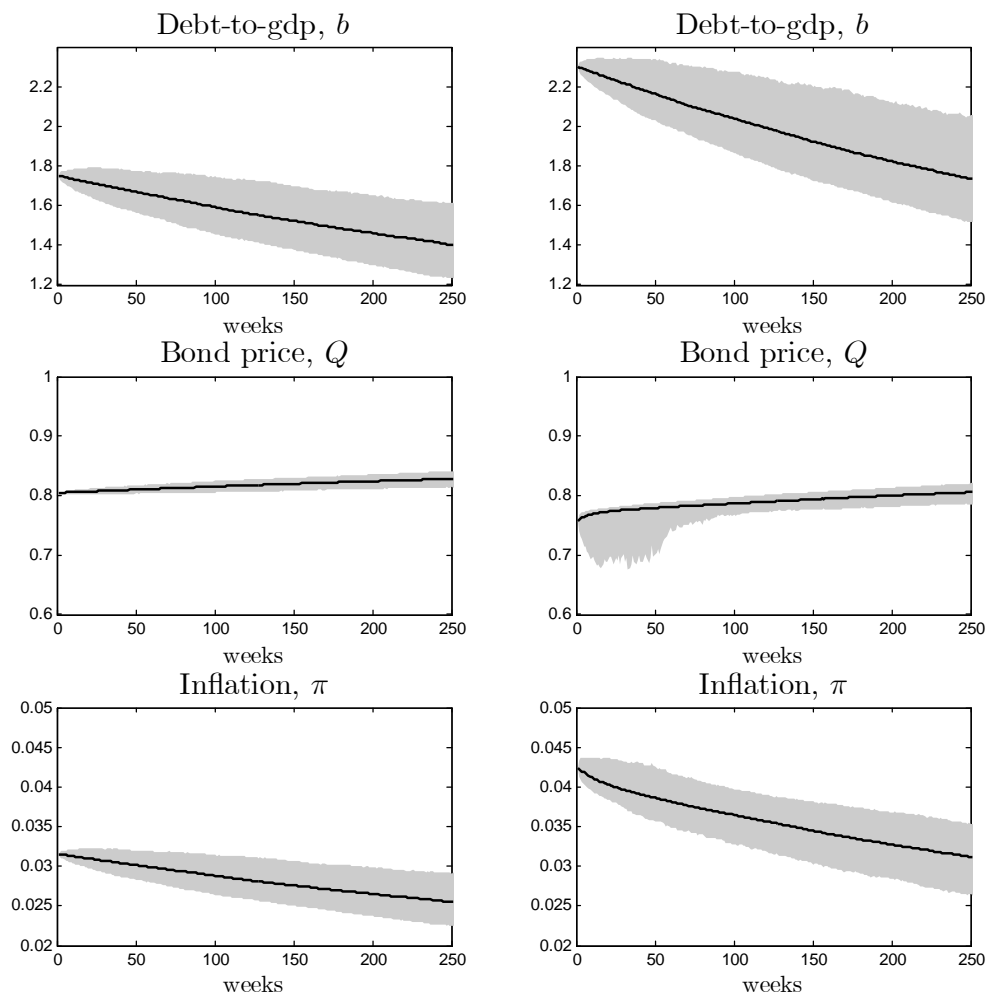


Figure 5: Dynamics under no-commitment policy. Initial state  $b_0 = 1.75$  (left column) and  $b_0 = 2.30$  (right column)

essentially reflect anticipated inflation. As regards the latter (lower panel), it starts at a higher level than in the left column (due of course to higher initial debt), but then follows a similar decline as debt, and hence the incentive to deflate it away, gradually falls.

These results yield a clear picture about the trade-offs between both monetary policy regimes and their impact on sovereign debt sustainability. As long as debt remains sufficiently far away from the default threshold, the anti-inflationary regime keeps inflationary distortions and bond yields at a minimum while posing no challenge to debt sustainability. This allows the anti-inflationary regime to achieve superior welfare outcomes, as shown by the value functions in Figure 1. However, if debt comes sufficiently close to default levels (e.g. due to a string of bad output shocks), investors start perceiving default as rather imminent, demanding bond yields that, not before long, precipitate such a default. By contrast, the no-commitment monetary policy makes debt sustainable even at levels that would trigger default under the anti-inflationary regime. Thanks to the possibility of creating inflation to stabilize the debt ratio, the no-commitment policy enjoys a margin of flexibility that avoids a collapse in bond prices and eventually default. At such debt levels, the benefits of making debt sustainable outweigh the costs associated to inflationary distortions and the rise in nominal bond yields, thus producing better welfare outcomes.

## 4 “I will do whatever it takes...”: announcing a loose monetary commitment

### 4.1 Description

**OMT.** In this section we go one step further and ask the following questions. Can an anti-inflationary Central Bank stabilize the debt path just by announcing that it will switch to a no-commitment strategy if sovereign spreads were to rise above a certain threshold? If so, which is the optimal threshold for this switch? These questions are at the basis of the Outright Monetary Transactions (OMT) programme announced by the ECB on 2nd August 2012 and preceded by the famous statement (on July 26th) by Mario Draghi, President of the ECB:

“Within our mandate the ECB is ready to do whatever it takes to preserve the euro, and believe me: it will be enough.”

The OMT allow the ECB to step in the secondary sovereign debt markets (in particular markets for sovereign bonds with a maturity of between one and three years) without any quantitative limit. They have never been implemented in practice. After the announcement there was a progressive reduction in the spreads of the sovereign debt of most Eurozone periphery countries.

**Model.** Within the context of the model outlined above, we model the OMT as follows: the Central Bank keeps inflation at zero but announces that, if it becomes optimal to do so, it will abandon its anti-inflationary commitment and switch permanently to the no-commitment regime.<sup>23</sup> This policy may thus be interpreted as a conditional, or loose, anti-inflationary commitment. The choice of when to activate the OMTs is a new optimal stopping problem, different from the optimal default choice studied in the previous section. We let  $\tilde{T}$  denote the smallest time, with respect to the filtration  $\{\mathcal{F}_t\}$ , to switch from the anti-inflationary to the no-commitment regime. The solution is characterized by a partitioning of the state space in three regions:

1. An *anti-inflationary* region  $\tilde{\Phi} \equiv [0, \tilde{b})$  in which the government does not default and the Central Bank sticks to the zero inflation commitment,  $\pi(b) = 0$ .
2. A *no-commitment* region  $\hat{\Phi} \equiv [\tilde{b}, b^*)$  in which the government does not default and the Central Bank chooses inflation optimally at each point in time.<sup>24</sup>
3. A *default* region  $[b^*, \infty)$  in which the government defaults, after which inflation is optimally zero.

Importantly, the threshold debt ratio  $\tilde{b}$  that triggers the switch to the no-commitment regime (henceforth the *OMT threshold*) is optimally determined, analogously to the default threshold  $b^*$ . Notice also that, while the ECB announcement implied a (non-explicit) threshold for bond spreads for the purpose of activating the OMT, the fact that bond prices decrease monotonically with the debt ratio implies that a debt-based threshold is equivalent to a bond price-based one:  $b \geq \tilde{b}$  if and only if  $\tilde{Q}(b) \leq \tilde{Q}(\tilde{b})$ , where  $\tilde{Q}$  is the bond price function under the OMT program.<sup>25</sup>

We thus have the following value function for the OMT case, given an initial debt ratio  $b$ :

$$\begin{aligned} \tilde{V}(b) &= \max_{\tilde{b}} E_0 \left[ \int_0^{\tilde{T}(\tilde{b})} e^{-\rho t} (\log(1 + c_t)) dt + e^{-\rho \tilde{T}(\tilde{b})} \tilde{V}(\tilde{b}) \mid b_0 = b \right], \quad \forall b \in [0, \tilde{b}), \\ \tilde{V}(b) &= V(b) = \max_{\pi, b^*} E_0 \left[ \int_0^{T(b^*)} e^{-\rho t} (\log(1 + c_t) - \psi \pi_t^2) dt + e^{-\rho T(b^*)} \underline{V} \mid b_0 = b \right], \quad \forall b \in [\tilde{b}, b^*), \end{aligned} \quad (19)$$

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<sup>23</sup>We assume that the policy switch is permanent for tractability. One could also extend the analysis to the case in which the central bank can switch back to the anti-inflationary regime if it finds it optimal to do so. We leave this analysis for future research.

<sup>24</sup>If  $\tilde{b} > b^*$ , the no-commitment region is empty ( $\hat{\Phi} = \emptyset$ ).

<sup>25</sup>We consider that the fiscal rule is the same in all the regions, that is, we ignore the potential moral hazard problem that may emerge if the government decides to deviate from its rule in the no-commitment region. We leave this interesting question for future research.

subject to debt dynamics (4).<sup>26</sup> Here  $\tilde{T}(\tilde{b})$  is the smallest time  $\tilde{T}$  such that  $b_{\tilde{T}} = \tilde{b}$ . Notice that for  $b \in [\tilde{b}, b^*)$  the value function is exactly the same as in the no-commitment case (equation 9), which we denoted as  $V(b)$ ; that is,  $\tilde{V}(b) = V(b)$  for all  $b \in [\tilde{b}, b^*)$ . The boundary conditions at the default threshold  $b^*$  are given again by (10) and (11). In addition, we have two new boundary conditions at the optimal OMT threshold  $\tilde{b}$ :

$$\begin{aligned}\tilde{V}(\tilde{b}) &= V(\tilde{b}), \\ \tilde{V}'(\tilde{b}) &= V'(\tilde{b}).\end{aligned}$$

That is, we have analogous value matching and smooth pasting conditions requiring that, at the optimal OMT boundary, the level and slope of the value function under OMT should equal those of the value function in the no-commitment case.

The bond price function  $\tilde{Q}(b)$  satisfies the following equations,

$$\begin{aligned}\tilde{Q}(b) &= E \left[ \int_0^{\tilde{T}(\tilde{b})} e^{-(\rho+\lambda)t} (\lambda + \delta) dt + e^{-(\rho+\lambda)\tilde{T}(\tilde{b})} Q(\tilde{b}) \right], \quad \forall b \in [0, \tilde{b}), \\ \tilde{Q}(b) &= Q(b) = E \left[ \int_0^{T(b^*)} e^{-(\rho+\lambda)t - \int_0^t \pi_s ds} (\lambda + \delta) dt + e^{-(\rho+\lambda)T(b^*) - \int_0^{T(b^*)} \pi_s ds} \theta \right], \quad \forall b \in [\tilde{b}, b^*).\end{aligned}$$

That is, as long as  $b < \tilde{b}$  the bond is priced with no inflation.<sup>27</sup> Finally, we have the boundary condition on the bond price function,

$$\tilde{Q}(\tilde{b}) = Q(\tilde{b}), \tag{20}$$

where  $Q(b)$  is the bond price function under no commitment. Equation (20) guarantees the continuity of the bond price schedule at the OMT threshold. The boundary condition for  $Q(b)$  is given again by (16).

**Numerical algorithm.** In order to numerically solve this problem we proceed as follows:

1. *No-commitment region.* Solve the no-commitment problem over the domain  $[0, b^*)$ . Obtain  $V(b)$  and  $Q(b)$ . Guess an OMT threshold  $\tilde{b}^0$ . Set  $n = 1$ .
2. *Anti-inflationary region.* Solve an anti-inflationary problem over the domain  $[0, \tilde{b}^{n-1})$ . The boundary conditions are  $\tilde{V}'(\tilde{b}^{n-1}) = V'(\tilde{b}^{n-1})$  for the value function and  $\tilde{Q}(\tilde{b}^{n-1}) = Q(\tilde{b}^{n-1})$  for the bond price. Obtain  $\tilde{V}^n(b)$  and  $\tilde{Q}^n(b)$ .

---

<sup>26</sup>The value function (19)  $\forall b \in [0, \tilde{b})$  is conditional on not having exercised the OMTs yet. If the OMTs have already been exercised then it is equal to (9). Notice the abuse of notation as  $V(b)$  here refers to the value function in the no-commitment case.

<sup>27</sup>There is also an abuse of notation as  $Q(b)$  refers to the bond price in the no-commitment monetary policy regime.

3. *Value matching.* Given  $\tilde{V}^n$  from step 2, compute  $\tilde{V}^n(\tilde{b}^{n-1})$ . If  $\tilde{V}^n(\tilde{b}^{n-1}) > V^n(\tilde{b}^{n-1})$ , then increase  $\tilde{b}^n$  and set  $n := n + 1$ . If  $\tilde{V}^n(\tilde{b}^{n-1}) < V^n(\tilde{b}^{n-1})$ , then decrease  $\tilde{b}^n$ . Proceed again to step 2 until the value matching condition  $\tilde{V}(\tilde{b}) = V(\tilde{b})$  is satisfied.

The resulting value function defined over the interval  $[0, b^*)$  is  $\tilde{V}(b)$ ,  $\forall b \in [0, \tilde{b})$ , and  $V(b)$ ,  $\forall b \in [\tilde{b}, b^*)$ . Each individual problem is again solved following the steps described in Appendix A.

## 4.2 Results

Figure 6 shows the value and policy functions under the OMT regime (dotted blue line), together with the anti-inflationary and no-commitment cases. As shown by the upper left plot, the OMT programme outperforms both the anti-inflationary and no-commitment policies over the whole range of debt-to-GDP ratios. The reason is simple: OMT combines the virtues of the two baseline monetary regimes. At relatively low debt ratios, for which the policy switch has not taken place yet, the OMT regime keeps current and expected inflation at zero, and thus, like the anti-inflationary commitment, it avoids inflationary distortions and an inflation premium in sovereign yields. But as debt reaches the default threshold associated to the anti-inflationary regime (red circle), OMT avoids the collapse in bond prices that takes place in the latter regime, as investors anticipate the imminent switch to the no-commitment policy. As shown by the upper right panel, this switch triggers a discontinuous jump in inflation and hence a drop in the bond price (lower left panel) to its level under no commitment. Notice that the optimal OMT threshold  $\tilde{b}$ , where the inflation jump takes place, is essentially coincident with (though higher than) the crossing point of the value functions of the anti-inflationary and no-commitment regimes.<sup>28</sup> Above the OMT threshold, the latter policy simply replicates the no-commitment outcome.<sup>29</sup>

We also investigate the model's distributional dynamics under OMT, following the Monte Carlo procedure described in the previous section. Figure 7 presents the median paths of debt, bond prices and inflation, together with their 95 percent confidence intervals, conditional on an initial debt ratio of 175 percent, which makes it directly comparable with the right column of Figure 4 (anti-inflationary policy) and the left column of Figure 5 (no-commitment policy). Notice first that, as shown by the upper bound of the confidence interval, in 97.5 percent of the simulations the government does not default; in fact, we have verified that it does not default in any of our simulations (within our simulation horizon). This is not surprising, given that OMT pushes the default threshold up to its level under no commitment, 239 percent.

<sup>28</sup>The OMT threshold is  $\tilde{b} = 1.8057$ . Given the equilibrium bond price schedule, the OMT debt threshold is equivalent to an implicit bond yield of 650 basis points.

<sup>29</sup>We have also calculated the expected time to default under OMT as a function of the current debt ratio. We find that the time to default function under OMT is indistinguishable from the one under no-commitment (green line) in Figure 3.

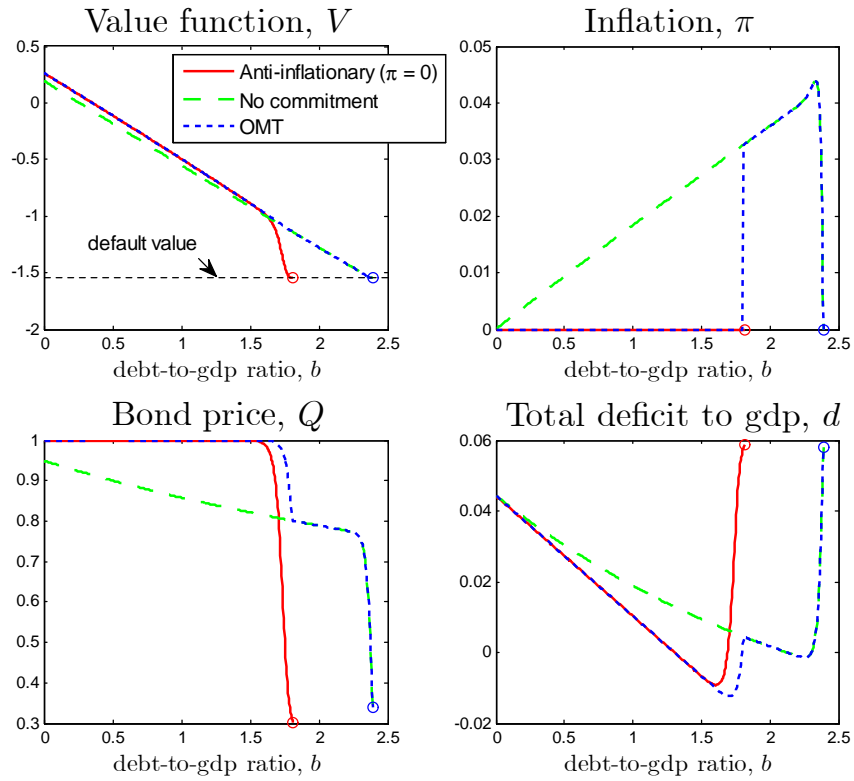


Figure 6: The effects of OMT on equilibrium

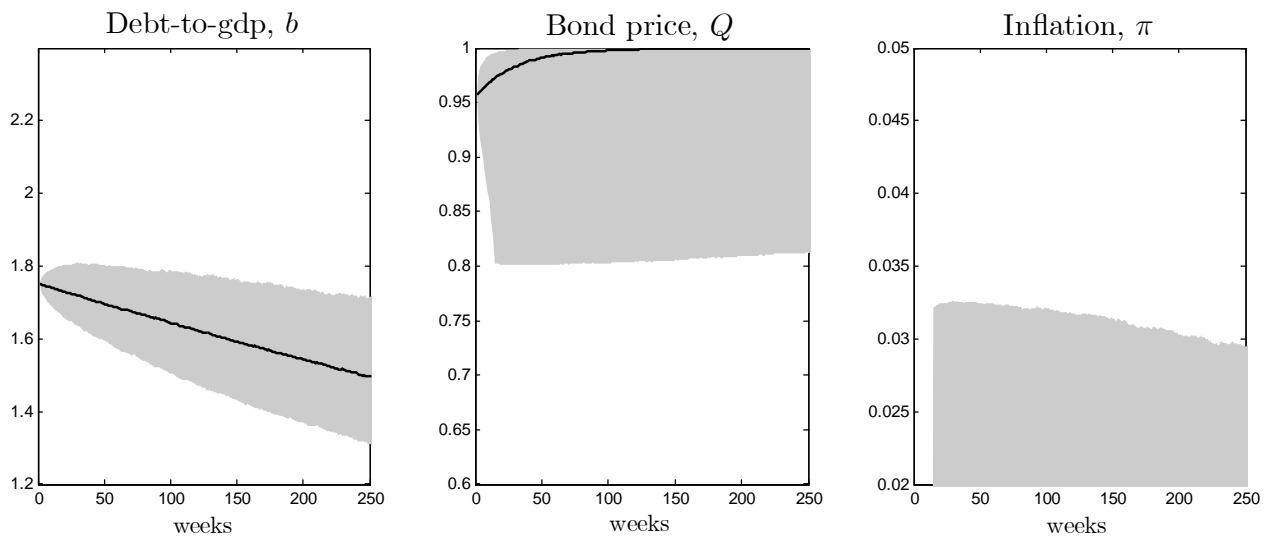


Figure 7: Dynamics under OMT. Initial value  $b_0 = 1.75$



As regards the bond price and inflation, in order to interpret the dynamics of their quantiles it is important to notice that OMT is activated in some simulations (in particular, in 17 percent of them) but not in others. As we saw in Figure 6, inflation jumps discontinuously from zero to 3.2 percent when OMTs are activated, whereas bond prices decline rapidly to their OMT boundary ( $Q(\tilde{b}) = 0.80$ ) right before the OMT threshold as investors anticipate the imminent inflation hike. Thus, the *upper* bound of the inflation confidence interval in Figure 7 reflects the jump in inflation that takes place in those simulations in which the OMTs *are* activated, which happens typically within a few weeks due to the proximity between  $b_0 = 1.75$  and the OMT threshold,  $\tilde{b} = 1.81$ . Likewise, the *lower* bound of the bond price confidence interval reflects the sharp decline in bond prices that takes place in those cases in which the switch to OMT takes place.<sup>30</sup>

To summarize, the OMT announcement optimally combines the virtues of the two baseline monetary regimes. For relatively low levels, for which the possibility of default is still remote, it avoids inflationary costs by sticking to the anti-inflationary commitment. But as debt approaches the level that would trigger default if such commitment was kept, the anticipation of its imminent abandonment reassures investors and avoids a collapse in bond prices. Once the policy switch takes place, the benefits from avoiding default once again outweigh the incurred inflation costs.

## 5 Exploring the mechanism

In this section we analyze the sensitivity of our baseline results to variations in some of the key model parameters. In particular, we will consider alternative calibrations for the response coefficient in the fiscal rule ( $\phi$ ), the debt amortization rate ( $\lambda$ ), and the scale of the inflation disutility cost function ( $\psi$ ).

### 5.1 Optimal monetary regime with a profligate government

We first analyze the effect of relaxing the 'toughness' of the fiscal rule, as captured by the response coefficient  $\phi$  of primary deficits to debt deviations from target. As explained in the calibration section, the coefficient  $\phi$  controls the half-life  $t^*$  of expected debt deviations.<sup>31</sup> Figure 8 displays the results, for the *no-commitment* case, for four different values of  $t^*$ : 12 (baseline), 20, 50 and 100 years.<sup>32</sup> Notice first that, in the baseline case with a relatively tight fiscal rule (grey line), the deficit ratio decreases with the debt ratio, even becoming negative for a small debt interval, and above a level around 230% it rises sharply as the burden of debt repayments starts dominating the stabilizing properties of the fiscal rule. However, as the rule becomes weaker (higher  $t^*$ ), the deficit

<sup>30</sup>Notice that the *median* path of inflation is zero.

<sup>31</sup>In particular,  $t^* = \log(2) / (\phi - \delta - \sigma^2 + \mu)$ .

<sup>32</sup>The corresponding to values of  $\phi$  are 0.074 (baseline), 0.051, 0.030, and 0.023, respectively.

ratio becomes higher and higher for debt ratios above  $\bar{b} = 60\%$ . The relaxation in the fiscal rule allows a higher primary deficit ratio  $c = (C - Y) / Y$  for intermediate debt ratios, which explains why the value function is higher in the case of the 20 years half-life (black line) than in the baseline for debt ratios over 30 percent of GDP. The resulting increase in consumption for medium and high debt ratios explains why the default threshold increases as the rule becomes weaker, reflecting a smaller cost in terms of foregone consumption.

It is however important to emphasize that, although lowering  $\phi$  increases the default threshold, this does *not* make debt more sustainable. Indeed, debt sustainability also depends on the deficit ratio, which determines the speed with which debt increases. In fact, it can be shown that lowering  $\phi$  *reduces* the expected time to default; that is, default thresholds are higher but they are reached earlier in expectation.<sup>33</sup> This explains why bond prices decrease faster with debt as the fiscal rule becomes weaker, which in turn explains the higher deficit ratios above 60% debt.

For sufficiently weak fiscal rules ( $t^* = 50, 100$ ), high deficits imply that debt starts spiralling out of control even while still substantially below the default threshold. Once this happens, deflating debt away becomes prohibitively costly and optimal inflation decreases rapidly towards zero. This pattern is particularly marked in the  $t^* = 100$  case (dotted dark blue line), which displays a value function below the others. In this case, the probability of a default is very high even for very low debt values, due to the government's extreme profligacy.<sup>34</sup>

Figure 9 compares the no commitment, anti-inflationary and OMT regimes for the case of  $t^* = 100$ . From a qualitative point of view the results are similar to those under the baseline calibration (fig. 6). The anti-inflationary regime dominates for low debt ratios, whereas the no-commitment regime is preferable for medium debt ratios. An important difference is that now the default threshold is essentially identical in both regimes (close to 320 percent). The reason is that both regimes yield the same value function for debt ratios above 220 percent. Intuitively, once debt starts spiralling out of control under a profligate fiscal rule, using inflation or not makes little difference for debt sustainability. Again, the OMT regime dominates the other two; the OMT threshold  $\tilde{b}$  is lower than in the baseline (140 percent, versus 181 percent), reflecting the fact that the value functions of the no-commitment and anti-inflationary regimes cross each other also at a lower debt ratio.

## 5.2 The impact of alternative debt maturities

We turn next to analyze the impact of changes in the debt maturity. In particular we analyze changes to the amortization rate  $\lambda$ , which controls the average debt duration  $\frac{1}{\rho + \lambda}$ . Figure 10 displays

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<sup>33</sup>These results are available upon request.

<sup>34</sup>For instance, the expected time to default for an initial debt ratio of 0 is just 500 years (not shown), compared to the 75.000 in the baseline

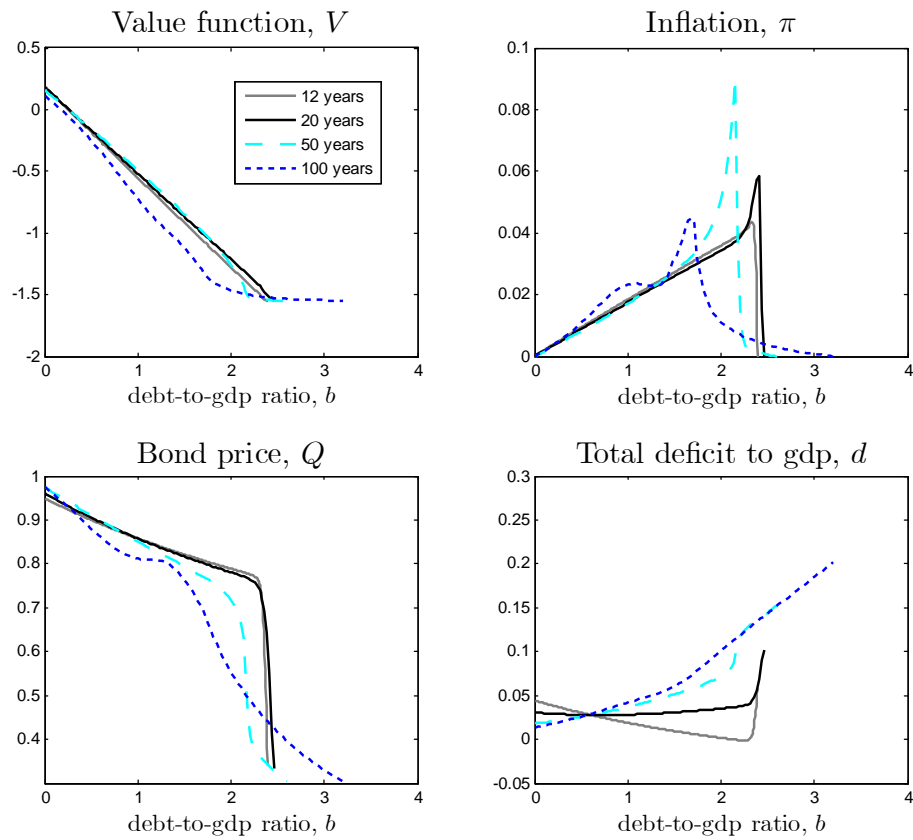


Figure 8: Equilibrium under no commitment for different fiscal policy rules

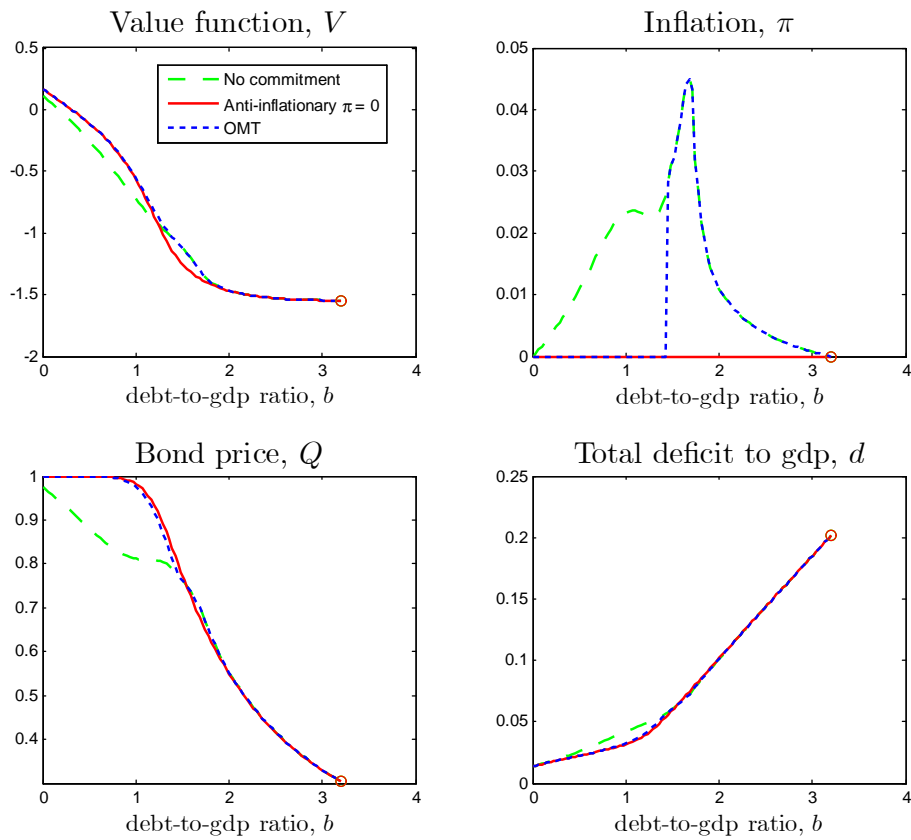


Figure 9: Equilibrium for alternative monetary regimes,  $t^* = 100$  years

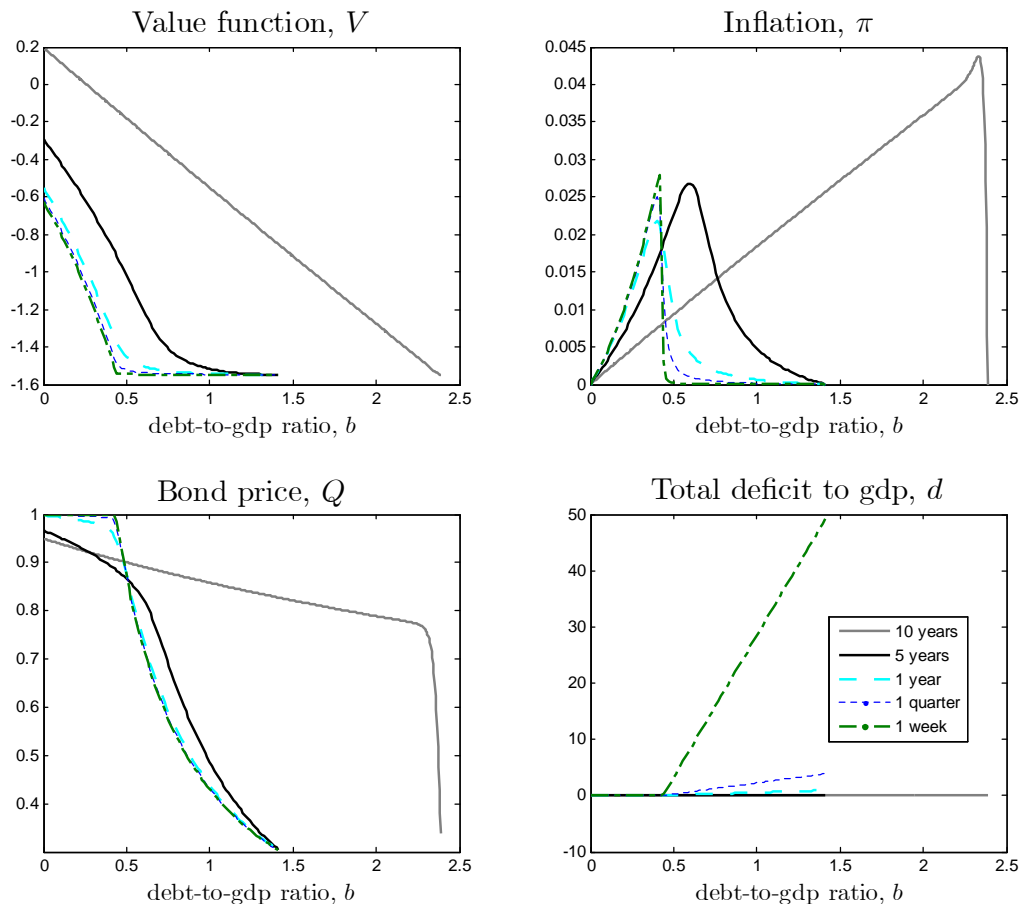


Figure 10: Equilibrium under no commitment for different debt durations

the results, again for the *no-commitment* regime, for average debt durations of 10 years (baseline), 5 years, 1 year, 1 quarter and 1 week.<sup>35</sup> We see that the reduction in the duration from 10 to 5 years produces a fall in the default threshold to around 140 percent, and it remains close to this value for shorter debt maturities. Notice how, as the duration tends to zero, the value function becomes almost linear with a kink. For debt values below the kink level, the price of the bond is close to 1 and the deficit is relatively small. However, for debt values above the kink, bond prices fall rather quickly and optimal inflation falls very rapidly to zero.<sup>36</sup>

Figure 11 compares the no commitment and anti-inflationary regimes for the case of one-quarter average debt maturity.<sup>37</sup> As in the baseline case, the anti-inflationary commitment dominates for low debt levels thanks to the absence of both inflationary distortions and a meaningful default risk.

<sup>35</sup>The corresponding values of  $\lambda$  are 0.06 (baseline), 0.16, 0.96, 3.96 and 49.96, respectively.

<sup>36</sup>None of these economies, except the baseline of course, has a deterministic stable equilibrium (not shown).

<sup>37</sup>One quarter is the (deterministic) duration assumed e.g. in Arellano's (2008) discrete-time model.

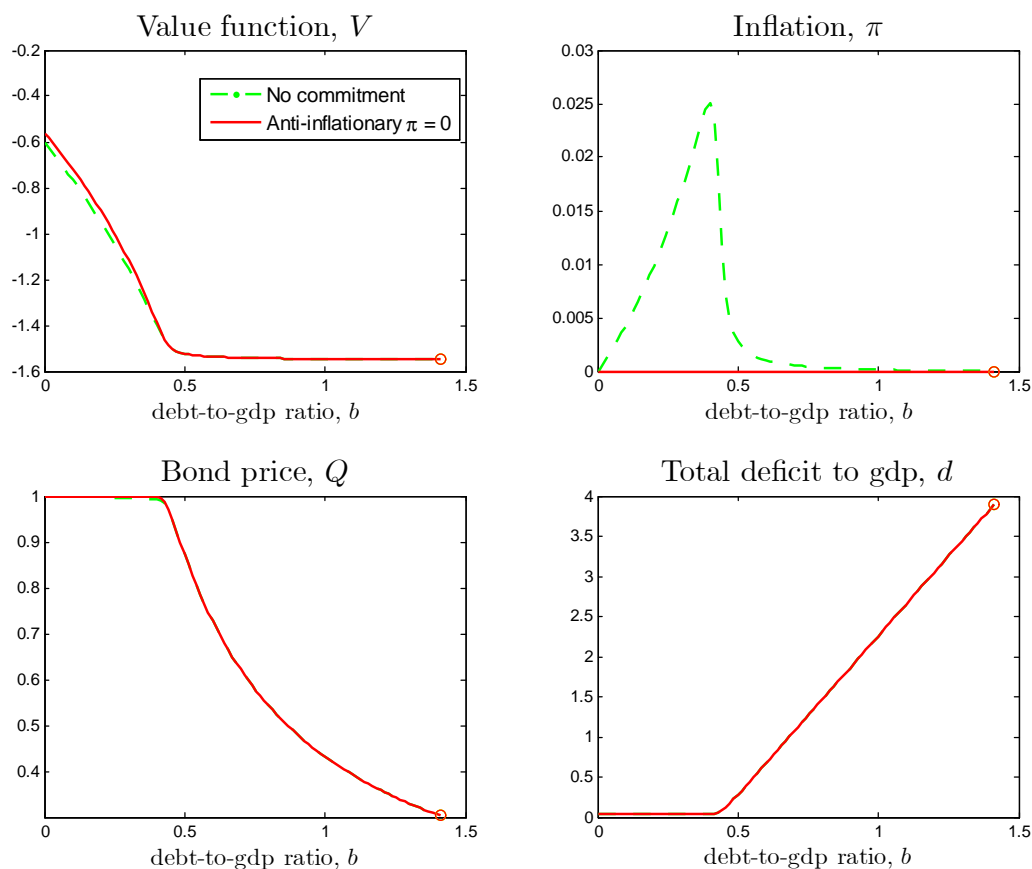


Figure 11: Equilibrium for alternative monetary regimes, one-quarter average debt duration

Both value functions again cross each other, at a level close to 45 percent. However, unlike in the baseline calibration, beyond the crossing point there is essentially no difference between both value functions. This in turn implies that both regimes have also the same default threshold. Intuitively, a very short debt maturity implies that the government must renew a large fraction of the debt stock within short intervals of time. Thus, the increase in nominal bond yields that would follow an attempt to deflate debt away would also affect a large fraction of the debt stock, thus making such a strategy prohibitively costly. As a result, optimal inflation never goes beyond moderate levels (2.5%) and returns very quickly to zero at debt levels substantially below the default threshold. Therefore, the OMT programme essentially replicates the anti-inflationary regime, as it dominates over the no-commitment one.

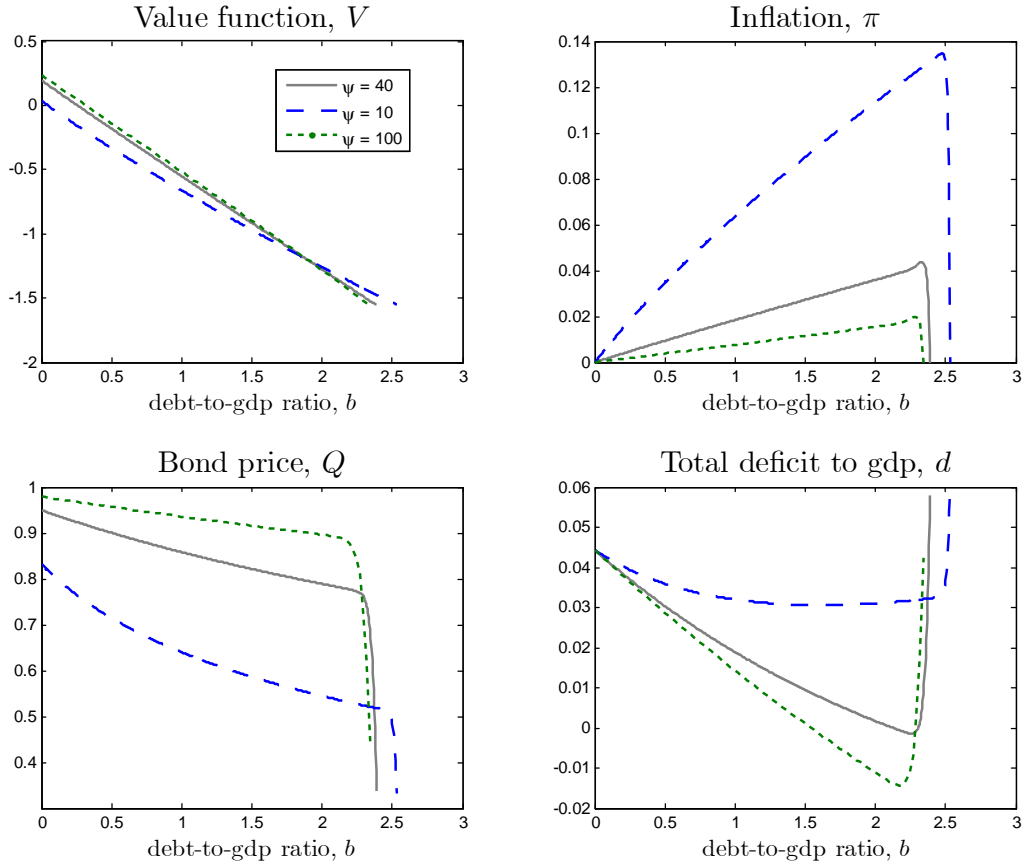


Figure 12: Equilibrium under no commitment for different inflation disutilities

### 5.3 Alternative inflation disutilities

Finally, we consider the effect of considering alternative scale parameters for inflation disutility. The baseline disutility parameter  $\psi$  was set to 40. Here we consider the cases of a looser and a tighter anti-inflationary stance, with  $\psi$  equal to 10 and to 100, respectively. Figure 12 displays the results for the no-commitment case. Not surprisingly, the maximum optimal inflation decreases with the inflation disutility parameter. Consequently, lower values of  $\psi$  also imply lower bond price schedules (except for debt ratios close to default), as investors demand a higher inflation premium. Also, the default threshold decreases with  $\psi$ . Intuitively, lower dislike of inflation means that *ceteris paribus* the central bank is more willing to deflate debt away so as to guarantee debt sustainability.

Figure 13 compares the no commitment, anti-inflationary and OMT regimes for the case of  $\psi = 10$ . There are no qualitative differences with respect to the baseline calibration. Even from a quantitative point of view the OMT threshold is very close to that in the baseline. The main

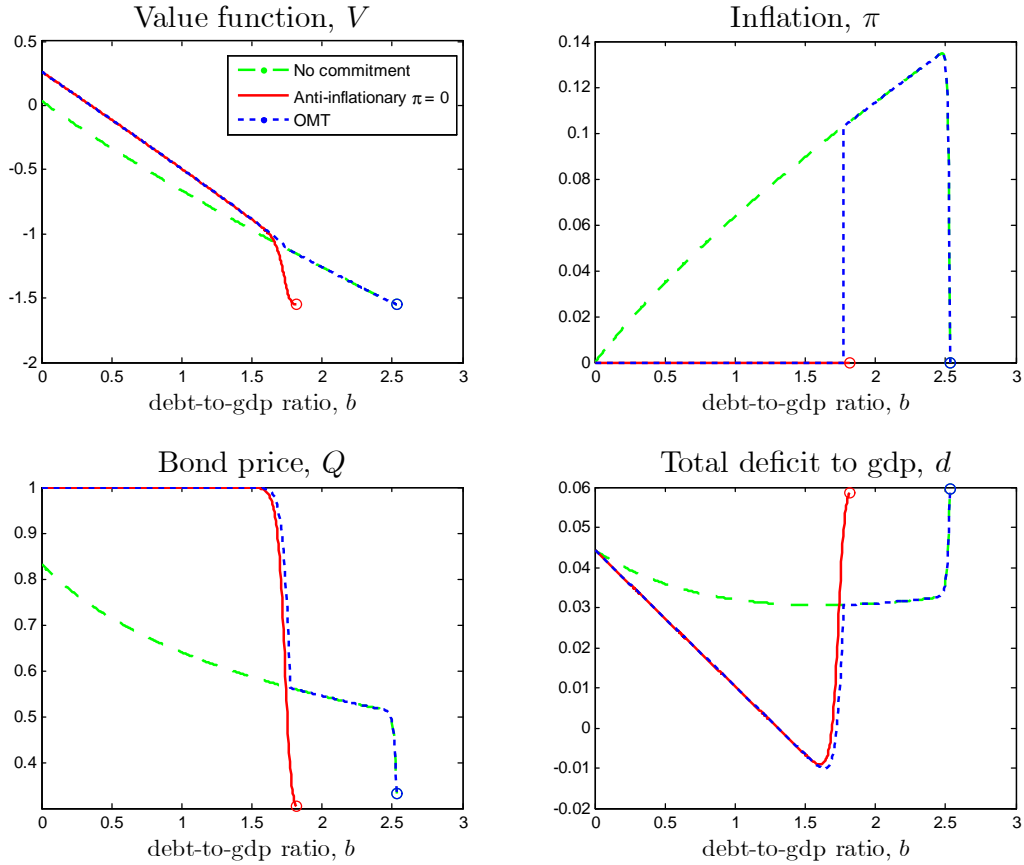


Figure 13: Equilibrium for alternative monetary regimes. Inflation disutility  $\psi = 10$

difference is the larger inflationary spike that takes place at the OMT threshold.

## 6 Conclusion

In this paper we investigate the effects of alternative monetary policy regimes on the sustainability of nominal sovereign debt, using a continuous-time open-economy model. We find that, when the debt-to-gdp ratio is relatively low, an anti-inflationary commitment is preferable to creating inflation so as to deflate debt away as it avoids inflationary distortions and an inflation premium in bond yields, while keeping default risks at a minimum. However, for debt ratios sufficiently close to default, it is optimal to abandon the anti-inflationary commitment, as this allows stabilizing debt and avoiding a collapse in bond prices.

We also study a monetary strategy in which the government announces a commitment to keeping inflation at zero unless the debt ratio surpasses a certain threshold, inspired by the ECB's OMT programme. We find that the OMT strategy dominates the two baseline regimes for any



debt ratio, by optimally combining their respective virtues.

In this paper we have not explored the possibility that the government may adapt its fiscal stance to changes in the monetary regime, e.g. as a result of moral hazard considerations. We view this as an important extension of our framework of analysis, which we leave for future research.

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# Appendix A: Description of the numerical algorithm

## Step 1: Solution to the Hamilton-Jacobi-Bellman equation

The HJB equation is solved by a finite difference scheme following Achdou et al. (2014). It approximates the value function  $V(b)$  on a finite grid with step  $\Delta b : b \in \{b_1, \dots, b_I\}$ . We use the notation  $V_i \equiv V(b_i)$ ,  $i = 1, \dots, I$ . The derivative of  $V$  with respect to  $b$  can be approximated with either a forward or a backward approximation:

$$V'(b_i) \approx \partial_F V_i \equiv \frac{V_{i+1} - V_i}{\Delta b}, \quad (21)$$

$$V'(b_i) \approx \partial_B V_i \equiv \frac{V_i - V_{i-1}}{\Delta b}, \quad (22)$$

$$V''(b_i) \approx \partial_{bb} V_i \equiv \frac{V_{i+1} + V_{i-1} - 2V_i}{(\Delta b)^2} \quad (23)$$

where the decision between one approximation or the other depends on the sign of the drift function

$$s_i = \left[ \left( \frac{\lambda + \delta}{Q(b_i)} + \sigma^2 - \mu - \lambda - \pi_i \right) b_i + \frac{c_i}{Q(b_i)} \right]$$

through an ‘‘upwind scheme’’ described below.

The HJB equation (12) is

$$\rho V(b) = \log(c+1) - \frac{\psi}{2} \pi^2 + \left[ \left( \frac{\lambda + \delta}{Q(b)} + \sigma^2 - \mu - \lambda - \pi \right) b + \frac{c}{Q(b)} \right] V'(b) + \frac{(\sigma b)^2}{2} V''(b),$$

where

$$\begin{aligned} c &= \phi(\bar{b} - b), \\ \pi &= -\frac{b}{\psi} V'(b), \end{aligned}$$

and smooth pasting boundary condition

$$V'(b^*) = \frac{V(b_{I+1}) - V(b_I)}{\Delta b} = 0 \Rightarrow V(b_{I+1}) = V(b_I).$$

The HJB is approximated by an upwind scheme

$$\frac{V_i^{n+1} - V_i^n}{\Delta} + \rho V_i^{n+1} = \log(c_i^n + 1) - \psi (\pi_i^n)^2 + \partial_F V_i^{n+1} s_{i,F}^n \mathbf{1}_{s_{i,F}^n > 0} + \partial_B V_i^{n+1} s_{i,B}^n \mathbf{1}_{s_{i,B}^n < 0} + \frac{(\sigma b_i)^2}{2} \partial_{bb} V_i^{n+1},$$

where

$$\begin{aligned} s_{i,F}^n &= \left( \frac{\lambda + \delta}{Q(b_i)} + \sigma^2 - \mu - \lambda + \frac{b_i}{\psi} \partial_F V_i^n \right) b_i + \frac{c_i}{Q(b_i)}, \\ s_{i,B}^n &= \left( \frac{\lambda + \delta}{Q(b_i)} + \sigma^2 - \mu - \lambda + \frac{b_i}{\psi} \partial_B V_i^n \right) b_i + \frac{c_i}{Q(b_i)}. \end{aligned}$$

The idea is that when the drift is positive ( $\mathbf{1}_{s_{i,F}^n > 0}$ ) we employ a forward approximation  $\partial_F V_i^{n+1}$  and when it is negative ( $\mathbf{1}_{s_{i,B}^n < 0}$ ) we employ a backward approximation  $\partial_B V_i^{n+1}$ . The term  $\frac{V_i^{n+1} - V_i^n}{\Delta} \rightarrow 0$  as  $V_i^{n+1} \rightarrow V_i^n$ . Moving all variables with  $n + 1$  superscripts to the left hand side and those with  $n$  superscripts to the right hand side:

$$V_{i-1}^{n+1} \alpha_i^n + V_i^{n+1} \beta_i^n + V_{i+1}^{n+1} \xi_i^n = \log(c_i^n + 1) - \frac{\psi}{2} (\pi_i^n)^2 + \frac{V_i^n}{\Delta}, \quad (24)$$

where

$$\begin{aligned} c_i^n &= \phi(\bar{b} - b_i), \\ \alpha_i^n &= \frac{s_{i,B}^n \mathbf{1}_{s_{i,B}^n < 0}}{\Delta b} - \frac{(\sigma b_i)^2}{2(\Delta b)^2}, \\ \beta_i^n &= \frac{1}{\Delta} + \rho + \frac{s_{i,F}^n \mathbf{1}_{s_{i,F}^n > 0}}{\Delta b} - \frac{s_{i,B}^n \mathbf{1}_{s_{i,B}^n < 0}}{\Delta b} + \frac{(\sigma b_i)^2}{(\Delta b)^2}, \\ \xi_i^n &= -\frac{s_{i,F}^n \mathbf{1}_{s_{i,F}^n > 0}}{\Delta b} - \frac{(\sigma b_i)^2}{2(\Delta b)^2}. \end{aligned}$$

The optimal inflation in the case with commitment is set to  $\pi_i^n = 0, \forall i$ . In the case without commitment it is set to

$$\pi_i^n = -\frac{b}{\psi} \left( \partial_F V_i \mathbf{1}_{s_{i,F}^n > 0} + \partial_B V_i \mathbf{1}_{s_{i,B}^n < 0} \right) + \mathbf{1}_{s_{i,F}^n \leq 0} \mathbf{1}_{s_{i,B}^n \geq 0} \bar{\pi}_i, \quad (25)$$

where  $\bar{\pi}_i$  is such that  $s_i = 0$ :

$$\bar{\pi}_i = \left( \frac{\lambda + \delta}{Q(b_i)} + \sigma^2 - \mu - \lambda \right) b_i + \frac{c_i}{b_i Q(b_i)}.$$

Equation (24) is a system of  $I$  linear equations which can be written in matrix notation as:

$$\mathbf{A}^n \mathbf{V}^{n+1} = \mathbf{d}^n, \quad (26)$$

where the matrix  $\mathbf{A}^n$  and the vectors  $\mathbf{V}^{n+1}$  and  $\mathbf{d}^n$  are defined by:

$$\mathbf{A}^n = \begin{bmatrix} \beta_1^n & \xi_1^n & 0 & 0 & \cdots & 0 \\ \alpha_2^n & \beta_2^n & \xi_2^n & 0 & \cdots & 0 \\ 0 & \alpha_3^n & \beta_3^n & \xi_3^n & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & \alpha_{I-1}^n & \beta_{I-1}^n & \xi_{I-1}^n \\ 0 & 0 & \cdots & 0 & \alpha_I^n & \beta_I^n + \xi_I^n \end{bmatrix},$$

$$\mathbf{V}^{n+1} = \begin{bmatrix} V_1^{n+1} \\ V_{2,1}^{n+1} \\ V_{3,1}^{n+1} \\ \vdots \\ V_{I-1}^{n+1} \\ V_I^{n+1} \end{bmatrix}, \quad \mathbf{d}^n = \begin{bmatrix} \log(c_1^n + 1) - \frac{\psi}{2} (\pi_1^n)^2 + \frac{V_1^n}{\Delta} \\ \log(c_2^n + 1) - \frac{\psi}{2} (\pi_2^n)^2 + \frac{V_2^n}{\Delta} \\ \log(c_3^n + 1) - \frac{\psi}{2} (\pi_3^n)^2 + \frac{V_3^n}{\Delta} \\ \vdots \\ \log(c_{I-1}^n + 1) - \frac{\psi}{2} (\pi_{I-1}^n)^2 + \frac{V_{I-1}^n}{\Delta} \\ \log(c_I^n + 1) - \frac{\psi}{2} (\pi_I^n)^2 + \frac{V_I^n}{\Delta} \end{bmatrix}.$$

Notice that the lower right element in  $A$  is  $\beta_I^n + \xi_I^n$  due to the smooth pasting condition ( $V_I^{n+1} = V_{I+1}^{n+1}$ ) and that the state constraint  $b > 0$  means that  $\alpha_1^n = 0$ .

The algorithm to solve the HJB equation runs as follows. Begin with an initial guess  $V_i^0 = -b_i$ , set  $n = 0$ . Then:

1. Compute  $\partial_F V_i^n$ ,  $\partial_B V_i^n$  and  $\partial_{bb} V_i^n$  using (21)-(23).
2. Compute  $\pi_i^n$  using (25).
3. Find  $V_i^{n+1}$  solving the linear system of equations (26).
4. If  $V_i^{n+1}$  is close enough to  $V_i^n$ , stop. If not set  $n := n + 1$  and go to 1.

## Step 2: Solution to the Bond Pricing Equation

The pricing equation (15) is also solved using an upwind finite difference scheme. The equation in this case is

$$Q(b) (\rho + \pi(b) + \lambda) = (\lambda + \delta) + \left[ \left( \frac{\lambda + \delta}{Q(b)} + \sigma^2 - \mu - \lambda - \pi(b) \right) b + \frac{c(b)}{Q(b)} \right] Q'(b) + \frac{(\sigma b)^2}{2} Q''(b), \quad (27)$$

with a boundary condition

$$Q(b^*) = \theta. \quad (28)$$

This case is similar to the HJB equation. Using the notation  $Q_i = Q(b_i)$ , the equation can be expressed as

$$\frac{Q_i^{n+1} - Q_i^n}{\Delta} + Q_i^{n+1}(\rho + \pi_i + \lambda) = \lambda + \delta + \partial_F Q_i^{n+1} s_{i,F}^n \mathbf{1}_{s_{i,F}^n > 0} + \partial_B Q_i^{n+1} s_{i,B}^n \mathbf{1}_{s_{i,B}^n < 0} \quad (29)$$

$$+ \frac{(\sigma b_i)^2}{2} \partial_{bb} Q_i^{n+1}, \quad (30)$$

where:

$$Q'(b_i) \approx \partial_F Q_i \equiv \frac{Q_{i+1} - Q_i}{\Delta b}, \quad (31)$$

$$Q'(b_i) \approx \partial_B Q_i \equiv \frac{Q_i - Q_{i-1}}{\Delta b}, \quad (32)$$

$$Q''(b_i) \approx \partial_{bb} Q_i \equiv \frac{Q_{i+1} + Q_{i-1} - 2Q_i}{(\Delta b)^2} \quad (33)$$

and rearranging terms

$$Q_{i-1}^{n+1} \alpha_i^n + Q_i^{n+1} (\beta_i^n + \pi_i + \lambda) + Q_{i+1}^{n+1} \xi_i^n = \lambda + \delta + \frac{Q_i^n}{\Delta}, \quad \forall i < i^*, \quad (34)$$

$$Q_i = \theta, \quad \forall i \geq i^*. \quad (35)$$

Notice the abuse of notation, as

$$s_{i,F}^n = s_{i,B}^n = s_i^n = \left( \frac{\lambda + \delta}{Q^n(b_i)} + \sigma^2 - \mu - \lambda - \pi_i \right) b_i + \frac{c_i}{Q^n(b_i)}.$$

Equation (29) is again a system of  $I$  linear equations which can be written in matrix notation as:

$$\mathbf{F}^n \mathbf{Q}^{n+1} = \mathbf{f}^n, \quad (36)$$

where the matrix  $\mathbf{F}^n$  and the vectors  $\mathbf{Q}^{n+1}$  and  $\mathbf{f}^n$  are defined by:

$$\mathbf{F}^n = \begin{bmatrix} \alpha_1^n + \beta_1^n + \pi_1 + \lambda & \xi_1^n & 0 & 0 & \cdots & 0 \\ \alpha_2^n & \beta_2^n + \pi_2 + \lambda & \xi_2^n & 0 & \cdots & 0 \\ 0 & \alpha_3^n & \beta_3^n + \pi_3 + \lambda & \xi_3^n & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & \alpha_{I-1}^n & \beta_{I-1}^n + \pi_{I-1} + \lambda & \xi_{I-1}^n \\ 0 & 0 & \cdots & 0 & \alpha_I^n & \beta_I^n + \pi_I + \lambda \end{bmatrix},$$

$$\mathbf{Q}^{n+1} = \begin{bmatrix} Q_1^{n+1} \\ Q_{2,1}^{n+1} \\ Q_{3,1}^{n+1} \\ \vdots \\ Q_{I-1}^{n+1} \\ Q_I^{n+1} \end{bmatrix}, \quad \mathbf{f}^n = \begin{bmatrix} \lambda + \delta + \frac{Q_1^n}{\Delta} \\ \lambda + \delta + \frac{Q_2^n}{\Delta} \\ \lambda + \delta + \frac{Q_3^n}{\Delta} \\ \vdots \\ \lambda + \delta + \frac{Q_{I-1}^n}{\Delta} \\ \lambda + \delta + \frac{Q_I^n}{\Delta} - \xi_I^n \theta \end{bmatrix}.$$

The algorithm to solve the bond pricing equation is similar to the HJB. Begin with an initial guess  $Q_i^0 = \frac{\lambda + \delta}{\rho + \lambda}$ , set  $n = 0$ . Then:

1. Find  $Q_i^{n+1}$  solving the linear system of equations (36).
2. If  $Q_i^{n+1}$  is close enough to  $Q_i^n$ , stop. If not set  $n := n + 1$  and go to 1.

## Appendix B: Computing the expected time to default

Let  $\tau(b)$  be the expected time to default given the current debt-to-GDP ratio  $b$ . It is defined as

$$\tau(b) = E[T] = E \left[ \int_0^T dt \right],$$

where  $T$  is the  $\mathcal{F}_t$ -adapted optimal time to default. Applying Itô's lemma we have

$$\tau(b) = \tau(b) + dt + \left[ \left( \frac{\lambda + \delta}{Q(b)} + \sigma^2 - \mu - \lambda - \pi \right) b + \frac{c}{Q(b)} \right] \tau'(b) dt + \frac{(\sigma b)^2}{2} \tau''(b) dt,$$

and so the expected time to default should satisfy

$$1 + \left[ \left( \frac{\lambda + \delta}{Q(b)} + \sigma^2 - \mu - \lambda - \pi \right) b + \frac{c}{Q(b)} \right] \tau'(b) + \frac{(\sigma b)^2}{2} \tau''(b) = 0,$$

with a boundary condition

$$\tau(b^*) = 0.$$