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Dynamic Efficiency Measurement: A Directional Distance Function Approach

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Summary
A dynamic directional input distance function is proposed within the adjustment-cost model of the firm, generalizing the directional distance function developed by Chambers, Chung and Färe (1996) in the static context. Duality is established between the dynamic directional distance function and the optimal current value function of the intertemporal cost minimization problem. Dynamic input efficiency measures are generated from the dynamic directional input distance function and intertemporal duality. These efficiency measures indicate the firm's degree of efficiency at a point along its adjustment path. The dynamic efficiency measures are illustrated on panel data of Dutch glasshouse horticulture firms using a nonparametric approach.

Keywords and Phrases: adjustment-cost technology, dynamic directional input distance function, dynamic duality, dynamic cost efficiency.

***JEL Classification Numbers*:** D21, D24, D61, D92.

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Introduction

The adjustment-cost model of the firm is an intertemporal (dynamic) approach to the theory of the firm where adjustment costs associated with changes in the level of the quasi-fixed factors are the source of the time interdependence of the firm's production decisions (e.g., Lucas 1967; Treadway 1969, 1970; Rothschild 1971; Mortensen 1973). Harmemesh and Pfann (1996) present an interesting survey of the literature on adjustment costs. The adjustment-cost model of the firm has been widely used in empirical work (e.g., Luh and Stefanou 1993, 1996; Nielsen and Schiantarelli 2003; Letterie and Pfann 2007; Letterie, Pfann and Verick 2010). However, primal and dual analytical foundations of the production theory with adjustment costs have not yet been explored as in the static theory of production.

Several primal representations of the production technology are defined and characterized axiomatically in the static theory of production, namely the production sets and the Shephard's distance functions (e.g., Shephard 1970; Debreu 1959; McFadden 1978; Färe and Primont 1995). Several generalizations of Shephard's distance functions have emerged in the production literature allowing extensions of the Farrell efficiency measures in the static context (e.g., Färe, Grosskopf and Lovell 1985, Chapters 5-7; Färe, Grosskopf and Lovell 1994, Chapter 8; Briec 1997; Bogetoft and Hougaard 1998; Chambers, Chung and Färe 1996, 1998; Färe and Grosskopf 2000a, 2000b; Chavas and Cox 1999; Halme *et al*. 1999). Specifically, the directional distance functions approach has guided recently much of the development in efficiency and productivity analysis (e.g., Chambers 2002, 2008; Ball *et al*. 2002a, 2002b; Färe *et al*. 2005).

In contrast, primal representations of the production technology in the context of the adjustment-cost theory of the firm have not yet been explored as in the static theory of the firm. The production function has been used, in general, as the primal representation of the adjustment-cost production technology (e.g., Epstein 1981; Lasserre and Ouellette 1999; Ouellette and Vigeant 2001). Recently, other primal representations of the adjustment-cost production technology have emerged in the literature allowing for the possibility of multiple outputs. Silva and Stefanou (2003) show that an adjustment-cost production technology can be represented by a family of input requirement sets satisfying some regularity conditions. A hyperbolic input distance function is defined in Silva and Stefanou (2007) to represent a production technology with adjustment costs and develop dynamic measures of production efficiency. In this paper, a directional input distance function is defined and characterized to represent an adjustment-cost production technology. The adjustment-cost (dynamic) directional input distance function generalizes the directional input distance function developed by Chambers, Chung and Färe (1996) in the static context.

Static duality is well established in the production theory: duality between production sets and optimal value functions (e.g., Shephard 1970; McFadden 1978; Färe and Primont 1995); duality between Shephard's distance functions and optimal value functions (e.g., Shephard 1970; Färe and Primont 1995); duality between directional distance functions and optimal value functions (Chambers, Chung and Färe 1996, 1998; Färe and Primont 2006). In contrast, intertemporal (dynamic) duality has been focused on the dual relation between the production function and the optimal value function of an intertemporal optimization problem (e.g., Epstein 1981; Lasserre and Ouellette 1999; Ouellette and Vigeant 2001), and duality between the optimal value function and the instantaneous variable profit function (McLaren and Cooper 1980). In the context of intertemporal cost minimization, this paper establishes duality between the adjustment-cost directional input distance function and the current value of the optimal value function.

Dynamic efficiency measurement is developed, in this paper, from the adjustment-cost directional input distance function and intertemporal duality. The dynamic directional input distance function provides difference measures of relative efficiency as opposed to radial measures (e.g., Nemoto and Goto 2003; Ouellette and Yan 2008) or hyperbolic measures as in Silva and Stefanou (2007).

This paper is structured as follows. In the next section, a directional input distance function is defined and characterized in the context of the adjustment-cost model of the firm. The dynamics are introduced in the production technology specification as an adjustment cost in the form of the properties of the directional input distance function with respect to the dynamic factors (or the change in the quasi-fixed factors). Section 3 establishes, in the context of intertemporal cost minimization, duality between the adjustment-cost directional input distance function and the current value of the optimal value function. Dynamic input-based efficiency measurement is discussed in section 4. Dynamic input inefficiency measures are generated from the adjustment-cost directional input distance function and duality between this function and the current value of the optimal value function. The empirical implementation of these inefficiency measures is illustrated using DEA techniques and some of these measures are applied to panel data of Dutch glasshouse horticulture firms in the period 1997-1999. The discussion of the DEA models is presented in section 5; the description of the data and the discussion of the empirical results are presented in section 6. The final section concludes.

Dynamic Directional Input Distance Function

The adjustment-cost production technology at time *t* is represented by a family of input requirement sets. The input requirement set is defined as (Silva and Stefanou 2003)

(1) $$V(y(t)|K(t)) = \{(x(t), I(t)) : (x(t), I(t)) \; can \; produce \; y(t) \; given \; K(t)\},$$

where $y(t) \in \Re^{M}_{++}$ is the vector of outputs, $x(t) \in \Re^{N}_{+}$ is the vector of variable inputs, $K(t) \in \Re^{F}_{++}$ is the capital stock vector and $I(t) \in \Re^{F}_{+}$ is the vector of gross investments (dynamic factors).

Including gross investment in the definition of $V(y(t)|K(t))$ implies maximum output levels not only depend on variable and quasi-fixed factors but also on the magnitude of the dynamic factors (change in the level of the quasi-fixed factors). Internal adjustment costs are incorporated in $V(y(t)|K(t))$ in the form of the properties of these sets with respect to the change in the quasi-fixed factors (see Silva and Stefanou 2003).

Properties of V:

V.1 $V(y(t)|K(t))$ is a closed and nonempty set.

V.2 $V(y(t)|K(t))$ has a lower bound.

V.3 If $(x(t), I(t)) \in V(y(t)\,|\,K(t))$ and $x(t)' \geq x(t)$, then $(x(t)', I(t)) \in V(y(t)\,|\,K(t))$.

V.4 If $(x(t), I(t)) \in V(y(t)\,|\,K(t))$ and $I(t)' \leq I(t)$, then $(x(t), I(t)') \in V(y(t)\,|\,K(t))$.

V.5 $V(y(t)|K(t))$ is a strictly convex set.

V.6 $K(t)' \geq K(t) \Rightarrow V(y(t)\,|\,K(t)) \subset V(y(t)\,|\,K(t)')$.

V.7 $y(t) \geq y(t)' \Rightarrow V(y(t)\,|\,K(t)) \subset V(y(t)'\,|\,K(t))$.

Some of these properties are the usual properties of input requirement sets in the static model of the firm: V.1-V.3 and V.7. The nonemptiness assumption of property V.1 implies feasibility and the closedness assumption of V.1 precludes that technology discontinuously changes from being able to produce *y* to not being able to produce *y*. Property V.3 establishes positive monotonicity of *V* in *x* implying additional units of any variable input increases *y*. Property V.7 asserts that outputs can be disposed of freely if necessary.

Properties V.4-V.6 are crucial to define the input requirement set in the context of the adjustment-cost model of the firm (Silva and Stefanou 2003). Property V.4 means that *V* is negative monotonic in *I*, implying there is a positive cost when investment in quasi-fixed factors takes place. This property reflects the presence of internal adjustment costs associated with gross investment. Property V.6 establishes that output levels are increasing in the stock of capital. Properties V.4 and V.6 together state that current changes in the dynamic factors decrease current levels of outputs but increase output levels in the future by increasing the future stocks of capital. Strict convexity in (*x*,*I*) (property V.5) leads to sluggish adjustment in the quasi-fixed factors since it implies an increasing marginal cost of adjustment. As shown below, strict convexity of *V* in (*x*,*I*), given *K* and *y*, implies strict concavity of the dynamic directional input distance function with respect to (*x*, *I*).

Definition 1. The dynamic directional input distance function $\vec{D}:\Re^{M}_{++}\times\Re^{F}_{++}\times\Re^{N}_{+}\times\Re^{F}_{+}\times\Re^{N}_{++}\times\Re^{F}_{++}\rightarrow\Re$ is defined as follows:

$\vec{D}(y(t),K(t),x(t),I(t);g_x,g_I)=\max\{\beta\in\Re:(x(t)-\beta g_x,I(t)+\beta g_I)\in V(y(t)\,|\,K(t))\}$, if $(x(t)-\beta g_x,I(t)+\beta g_I)\in V(y(t)\,|\,K(t))$ for some β and $-\infty$ otherwise.

$(g_x, g_I) \in \Re^N_{++} \times \Re^F_{++}$ is a nonzero vector determining the direction in which $\vec{D}$ is defined. This function measures the distance of ($x(t)$, $I(t)$) to the boundary of $V(y(t)|K(t))$ in a predefined direction $(g_x, g_I) \neq 0_{N+F}$. Given that βg_x is subtracted from $x(t)$ and βg_I is added to $I(t)$, this function is defined by simultaneously contracting variable inputs and expanding dynamic factors. Properties V.1 and V.2 of $V(y(t)|K(t))$ assure the maximization operation in definition 1 is well-defined.

Figure 1 illustrates the dynamic directional input distance function assuming one variable input and one dynamic factor. The input vector ($x(t)$,$I(t)$) is projected onto the isoquant of $V(y(t)|K(t))$ at a point $(x(t) - \vec{D}(.)g_x, I(t) + \vec{D}(.)g_I) \in V(y(t)|K(t))$, $(g_x, g_I) \neq 0_{N+F}$. Figure 1 shows three possible projections of the input vector ($x(t)$,$I(t)$) associated with three directions: g^0, g^1 and g^2.

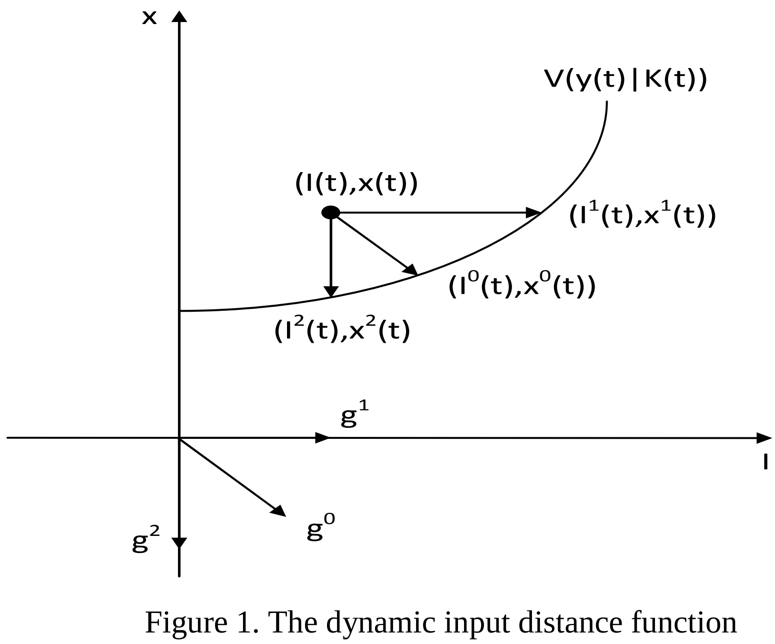

Figure 1. The dynamic input distance function

Using definition 1, the following relationship can be established

(2) $$\vec{D}(y(t), K(t), x(t), I(t); g_x, g_I) \geq 0 \Leftrightarrow (x(t), I(t)) \in V(y(t) \mid K(t)),$$

$(g_x, g_I) \in \Re^N_{++} \times \Re^F_{++}$. This relationship means that the dynamic directional input distance function represents fully the input requirement set. Thus, this function is an alternative primal representation of the adjustment-cost production technology.

***Lemma* 1**. $\vec{D}$ satisfies the following properties:

D.1 If V is strictly convex, $\vec{D}$ is strictly concave with respect to (x,I) given K and y.

D.2 $\vec{D}(y, K, x - \alpha g_x, I + \alpha g_I; g_x, g_I) = \vec{D}(y, K, x, I; g_x, g_I) - \alpha, \;\; \alpha \in \Re$.

D.3 If V.7, then $y' \geq y \Rightarrow \vec{D}(y', K, x, I; g_x, g_I) < \vec{D}(y, K, x, I; g_x, g_I)$.

D.4 If V.3, then $x' \geq x \Rightarrow \vec{D}(y, K, x', I; g_x, g_I) > \vec{D}(y, K, x, I; g_x, g_I)$.

D.5 If V.4, then $I' \leq I \Rightarrow \vec{D}(y, K, x, I'; g_x, g_I) > \vec{D}(y, K, x, I; g_x, g_I)$.

D.6 If V.6, then $K' \geq K \Rightarrow \vec{D}(y, K', x, I; g_x, g_I) > \vec{D}(y, K, x, I; g_x, g_I)$.

D.7 $\vec{D}(y, K, x, I; \alpha g_x, \alpha g_I) = \frac{1}{\alpha}\vec{D}(y, K, x, I; g_x, g_I)$, $\alpha > 0$.

D.8 $\vec{D}$ is continuous with respect to (x,I), given K and y.

The proof of Lemma 1 is presented in the Appendix. Properties D.2-D.4, D.7 and D.8 are analogous to the properties of the directional input distance function defined in the context of the static theory of production (see Chambers, Chung and Färe 1996). Property D.2 is the translation property; property D.7 states that the dynamic directional input distance function is homogeneous of degree (-1) in (g_x, g_I). Both of these properties result from definition 1. Property D.3 (D.4) states that the dynamic directional input function is decreasing (increasing) in y (x).

Properties D.5 and D.6 establish that the dynamic directional input distance function is, respectively, decreasing in *I* and increasing in *K*. These properties are inherited from the properties V.4 and V.6 of the input requirement set. Thus, properties D.5 and D.6 together imply that current changes in the dynamic factors decrease current levels of outputs but increase output levels in the future by increasing the future stocks of capital. Property D.1 results from property V.5 of the input requirement set and implies increasing marginal cost of adjustment leading to sluggish adjustment in the quasi-fixed factors.

Less restrictive properties of *V* can be assumed, resulting in less strict properties of the adjustment-cost directional input distance function. These properties were chosen to facilitate the characterization of duality in the next section.[1]

The Intertemporal Problem and Duality

Dynamic duality is a subject matter dating back to the papers of Cooper and McLaren (1980), McLaren and Cooper (1980), and Epstein (1981). Cooper and McLaren (1980) develop intertemporal duality in the context of the consumer theory. McLaren and Cooper (1980) and Epstein (1981) focus on intertemporal duality in the context of the adjustment-cost model of the firm. McLaren and Cooper (1980) establish intertemporal duality between the instantaneous variable profit function and the total profit function; Epstein (1981) establishes intertemporal duality between the total profit function and the production function. For a detailed analysis of the dynamic duality results developed by Epstein (1981), see Caputo (2005), chapter 20, pp. 537-558.

[1] Property V.5 can be defined in a less restrictive way, imposing convexity rather than strict convexity of the input requirement set. In this case, $\vec{D}$ is concave rather than strictly concave. Intertemporal duality, presented in the next section, can be established assuming a concave directional distance function. However, strict concavity of $\vec{D}$ eliminates the problem of multiple optimal solutions when establishing intertemporal duality.

More recently, Lasserre and Ouellette (1999) propose a duality theory in discrete time for an expected cost-minimizing firm in the presence of adjustment costs where the production technology is represented by a production function. Ouellette and Vigeant (2001) generalize the static duality between a cost function and a production function of a regulated firm to a dynamic context.

Our main goal in this section is to establish duality between $\vec{D}$ and the current value of the optimal value function of the intertemporal cost minimization problem. At any base period $t \in [0,+\infty)$, the firm is presumed to minimize the discounted flow of costs over time as follows

$$(3) \qquad \begin{aligned} W(y, K_t, w, c, r, \delta) = & \min_{(x(.), I(.))} \int_t^{+\infty} e^{-r(s-t)} \left[w' x(s) + c' K(s) \right] ds \\ & s.t. \\ & \dot{K}(s) = I(s) - \delta K(s), \ \ K(t) = K_t \\ & (x(s), I(s)) \in V(y(s) \mid K(s)), \ s \in [t, +\infty), \end{aligned}$$

where $y \in \Re^M_{++}$ is the output vector in the base period, $K_t \in \Re^F_{++}$ is the initial capital stock vector, $w \in \Re^N_{++}$ is the vector of rental prices of the variable input vector $x(s) \in \Re^N_{+}$, $c \in \Re^F_{++}$ is the vector of rental prices of the capital stock vector $K(s) \in \Re^F_{++}$. The vectors *w* and *c* represent current market prices (i.e., at *s* = *t*) that the firm expects to persist indefinitely. The vector $y \in \Re^M_{++}$ is the output vector in the base period that the firm expects to produce over time. This is the static price and output expectations hypothesis. The firm revises its price and output expectations as well as its production plans as the base period changes.

The discount rate is $r > 0$ and the δ is a diagonal $F \times F$ matrix of depreciation rates $\delta_f > 0$, $f = 1,\ldots,F$. The firm is assumed to have the same discount rate and the same depreciation matrix in all base periods to discount future costs and depreciate the capital stocks. Given this assumption, r and δ can be suppressed as arguments of the optimal value function W.

Given (2), the intertemporal cost minimization problem in (3) can be expressed as

$$
\text{(4)} \qquad
\begin{aligned}
W(y, K_t, w, c) = \min_{(x(.),I(.))} &\int_t^{\infty} e^{-r(s-t)} \left[w' x(s) + c' K(s) \right] ds \\
s.t.& \\
&\dot{K}(s) = I(s) - \delta K(s), \;\; K(t) = K_t \\
&\vec{D}(y(s), K(s), x(s), I(s); g_x, g_I) \geq 0, \;\; s \in [t, +\infty).
\end{aligned}
$$

In order to establish duality between $\vec{D}$ and W, additional assumptions are defined. These assumptions are analogous to the assumptions postulated in Epstein (1981) and allow us to use differential calculus to establish duality between $\vec{D}$ and W. Besides properties D.1-D.8, $\vec{D}$ is assumed to satisfy the following property: D.9 $\vec{D} \in C^{(1)}$ and $\vec{D}_l \in C^{(1)}$, $l = y, K, x, I$. In addition, the following conditions are assumed to hold:

(a.1) For each (y, K_t, w, c), there exists a unique solution for problem (4) in the sense of convergent integrals; the policy functions $x^*(y, K_t, w, c)$ and $I^*(y, K_t, w, c)$ are $C^{(1)}$ and the current value shadow price function $\theta^*(y, K_t, w, c)$ is $C^{(2)}$.

(a.2) For each (y, K_t, x^0, I^0), there exists (y, K_t, w^0, c^0) such that (x^0, I^0) is optimal for problem in (4) at $s = t$, given the vectors y and K_t and the rental prices vectors w and c.

Assumption D.9 guarantees smoothness conditions necessary to use differential calculus to establish duality between $\vec{D}$ and W. Assumption (a.1) establishes the existence of a unique and differentiable optimal solution to problem (4). Given problem (4), the only points (y, K_t, x^0, I^0) that matter are the ones satisfying condition (a.2).

Given properties D.1-D.9 of $\vec{D}$ and assumptions (a.1) and (a.2), the current value of the optimal value function *W* associated with problem in (4) obeys the Hamilton-Jacobi-Bellman (H-J-B) equation (e.g., Epstein (1981); Kamien and Schwartz 1991, section 21, pp. 259-263; Caputo 2005, chapter 19, pp. 511-532)[2]

(5) $rW(y,K,w,c) = \min_{x,I} \{w'x + c'K + W_K(y,K,w,c)(I - \delta K) : \vec{D}(y,K,x,I;g_x,g_I) \geq 0\}.$

The H-J-B equation is valid for any base period *t*, $t \in [0,+\infty)$. *K* is any possible capital vector in the base period and $W_K(y,K,w,c)'$ is the vector of the current shadow (or marginal) value of capital. By definition, the current shadow value of the quasi-fixed factor *f*, W_{K_f}, measures the impact on the optimal current value function due to a small change in the initial capital stock K_f. Therefore, the current shadow value of capital is an endogenous price influenced by the rental prices (*w*,*c*), the initial capital stocks and the vector of the production targets.

The H-J-B equation in (5) can be converted to the following unconstrained problem as

(6) $rW(y,K,w,c) = \min_{x,I} \{w'x + c'K + W_K(y,K,w,c)(I - \delta K) + \lambda \vec{D}(y,K,x,I;g_x,g_I)\},$

where $\lambda = W_K(.)g_I - w'g_x$. Proof of equation (6) is presented in the Appendix. λ is the firm´s current valuation of the directional vector, which is equal to the current shadow value of the dynamic factor direction minus the current market value of the variable input direction. Note that, along the optimal path, $\lambda \vec{D}(.) = 0$.

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[2] All vectors are column vectors and the derivative of a scalar-valued function with respect to a column vector is a row vector.

The H-J-B equation in (6) states that the total opportunity cost of the optimal input vector in the base period *t*, *rW*, is equal to the current total cost plus the current shadow value of the optimal net investments. One of the optimal conditions for an interior solution in problem (6) is the following: $W_{K_f} = -\lambda \vec{D}_{I_f}$, $f = 1,\dots,F$. These conditions show that the adjustment costs associated with changes in the quasi-fixed factors are the source of the time interdependence of the firm´s production decisions. Note that those conditions establish that the current shadow value of a unit of the quasi-fixed factor *f* is equal to its current marginal cost. As shown in the Appendix, the current shadow value of a unit of the quasi-fixed factor *f* equals the discounted stream of the net marginal benefits it generates from the base period to infinity

(7) $$W_{K_f}(.) = \int_t^{+\infty} e^{-r(s-t)} \left(-e^{-\delta_f (s-t)} c_f - e^{-\delta_f (s-t)} \lambda \vec{D}_{K_f}(.) \right) ds .$$

Given (7), those optimality conditions imply that the optimal investment decisions result from a balance between the current marginal cost and the discounted stream of the future net marginal benefits generated by an additional unit of each quasi-fixed factor.

The optimal values of (*x*, *I*) for problem (6), given (y, K, w, c), are equal to optimal values of the control variables in the intertemporal cost minimization problem in (4) when *s* = *t* (assumption (a.2)). By assumptions (a.1) and (a.2), for each (y, K, w, c), the optimal values of (*x*, *I*) in problem (6) are given by the values of the policy functions $x^*(y, K, w, c)$ e $I^*(y, K, w, c)$, which are the optimal values of the control variables in the intertemporal cost minimization problem in (4) in any base period *t*, $t \in [0, +\infty)$, given that the capital stock vector in the base period is *K*.

The H-J-B equation in (6) is important to establish duality between $\vec{D}$ and *W* since the problem in (6) is a static optimization problem relating these functions (e.g., Epstein 1981). Consequently, the static duality theory can be applied. Theorems 1 and 2 below establish intertemporal (dynamic) duality between $\vec{D}$ and *W*. The proofs of these theorems are presented in the Appendix.

Theorem 1: Let $\vec{D}$ satisfy properties D.1-D.9 and assume conditions (a.1) and (a.2). Define *W* as in problem (4). Then, *W* satisfies the following properties

W.1 *W* is a real-valued function; $W(.) \in C^{(2)}$ and $W_K(.) \in C^{(2)}$.

W.2 *W* is increasing in *y*.

W.3 *W* is decreasing in K_t.

W.4 (a) $W_{Ky}(.)'(I^* - \delta K) - rW_y' < 0^M$, (b) $W_{KK}(.)(I^* - \delta K) - (r + \delta)W_K(.)' + c > 0^F$,

(c) $W_{Kc}(.)'(I^* - \delta K) = rW_c(.)' - K$, (d) $W_{Kw}(.)'(I^* - \delta K) = rW_w(.)' - x^*$.

W.5 *W* is homogeneous of degree one in (*w*, *c*).

W.6 (a) *W* is increasing in *w*; (b) *W* is increasing in *c*.

W.7 *W* is concave in (*w*, *c*), given *K* and *y*.

W.8 For any (y, K, w, c), define the following problem

$$F(y, K, x, I; g_x, g_I) = \min_{w,c} \left\{ \frac{w'x + c'K + W_K(.)(I - \delta K) - rW(y, K, w, c)}{w'g_x - W_K(.)g_I} \right\},$$

$w'g_x - W_K g_I \neq 0$. (a) For (y, K, w^0, c^0), the minimum value in the previous problem occurs at $(w, c) = (w^0, c^0)$ if $(x, I) = (x^*(y, K, w^0, c^0), I^*(y, K, w^0, c^0))$. (b) *F* is non-negative and strictly concave in (*x*, *I*), given *y* and *K*.

Theorem 1 establishes that *W* is obtained from $\vec{D}$. The meaning and implications of the properties of the optimal value function *W* can be deduced from the proof of theorem 1. Before presenting theorem 2, some of those properties are analyzed. Property W.3 implies that $W_{K_f} < 0, f = 1, ..., F$, and is dual to property D.5, which, in turn, implies that $D_{I_f} < 0, f = 1,...,F$. Properties W.4(a) and W.4(b) imply restrictions on the optimal value function. In particular, property W.4(b) is dual to the property D.6, which implies that $D_{K_f} > 0, f = 1,...,F$. An intertemporal version of the Shephard's lemma can be constructed from properties W.4(c) and W.4(d). Property W.8 establishes that $F(y,K,x,I;g_x,g_I) = \vec{D}(y,K,x,I;g_x,g_I)$ and this relation is important to construct theorem 2.

Theorem 2: Let W satisfy properties W.1-W.8. Define

$$\vec{D}(y,K,x,I;g_x,g_I) = \min_{w,c}\left\{\frac{w'x + c'K + W_K(.)(I - \delta K) - rW(y,K,w,c)}{w'g_x - W_K(.)g_I}\right\},$$

$w'g_x - W_K g_I \neq 0$, $(g_x,g_I) \in \Re^N_{++} \times \Re^F_{++}$, $(g_x,g_I) \neq 0_{N+F}$. Then, over its domain of definition, $\vec{D}$ satisfies properties D.1-D.9.

Theorem 2 establishes that it is possible to recover $\vec{D}$ from the current value of the optimal value function. The objective function in Theorem 2 is equal to the difference between the total opportunity cost of the input vector (*x*,*I*) and the minimum total opportunity cost, normalized by the firm´s valuation of the directional vector.

Theorems 1 and 2 prove the existence of the following duality between the dynamic directional input distance function and the current value of the optimal value function:

(8a)
$$\begin{aligned} rW(y,K,w,c) = \min_{x,I}\{&w'x + c'K + W_K(y,K,w,c)(I - \delta K) \\ &+ (W_K(y,K,w,c)g_I - w'g_x)\vec{D}(y,K,x,I;g_x,g_I)\},\end{aligned}$$

(8b)
$$\vec{D}(y,K,x,I;g_x,g_I) = \min_{w,c}\left\{\frac{w'x + c'K + W_K(.)(I - \delta K) - rW(y,K,w,c)}{w'g_x - W_K(.)g_I}\right\},$$

$w'g_x - W_K g_I \neq 0$.

The optimal value function for the minimization problem in (8a), defined in the input space, is rW, with policy functions $x^*(y, K_t, w, c)$ and $I^*(y, K_t, w, c)$. The dynamic directional input distance function is the optimal value function for the minimization problem in (8b), defined in the rental prices space, with optimal functions $w^*(y, K, x, I)$ and $c^*(y, K, x, I)$. Note that $x = x^*(y, K, w, c)$ and $I = I^*(y, K, w, c)$ if and only if $w^*(y, K, x, I) = w$ and $c^*(y, K, x, I) = c$.

Dynamic Efficiency Measurement

From (8a) and (8b), we may write

(9) $rW(y, K, w, c) \leq w'x + c'K + W_K(I - \delta K) + (W_K g_I - w'g_x)\vec{D}(y, K, x, I; g_x, g_I)$.

This inequality can be rearranged as

(10) $$OE = \frac{w'x + c'K + W_K(.)(I - \delta K) - rW(y, K, w, c)}{w'g_x - W_K(.)g_I} \geq \vec{D}(y, K, x, I; g_x, g_I),$$

where the left-hand side is the dynamic cost inefficiency measure (OE). This measure is the normalized deviation between the total shadow cost of the actual choices and the minimum total shadow cost. The normalization is the firm's valuation or shadow value of the direction vector, making the dynamic cost inefficiency a unit-free measure. The right-hand side is the dynamic directional input distance function measure of technical inefficiency of variable and dynamic factors.

Expression (10) can be modified by introducing allocative inefficiency (AE), rendering it as the following equality,

(11) $$OE = \frac{w'x + c'K + W_K(.)(I - \delta K) - rW(y, K, w, c)}{w'g_x - W_K(.)g_I} = \vec{D}(y, K, x, I; g_x, g_I) + AE,$$

with $AE \geq 0$. Note that the dynamic cost inefficiency measure (and the dynamic directional input distance function) depends on the direction vector chosen, (g_x, g_I). One possible choice is $(g_x, g_I) = (x, I)$, i.e., the direction vector is equal to the observed variable input vector and the observed dynamic factor vector.

A directional distance function for dynamic factors can be derived as a special case of the dynamic directional input distance function in definition 1:

(12) $$\vec{D}_I(y, K, x, I; 0_N, g_I) = \max\{\beta_I : (x, I + \beta_I g_I) \in V(y \mid K)\},$$

with $g_x = 0_N$ and $\vec{D}_I(y, K, x, I; 0_N, g_I) \geq 0$. Properties of $\vec{D}_I$ are derived from the properties of $\vec{D}$ (e.g., strict concavity in *I*, increasing in *x*). The directional distance function in (12) provides a measure of technical inefficiency of the dynamic factors of production.

Using (11) and (12), the shadow cost inefficiency of dynamic factors can be expressed as

(13) $$OE_I = \frac{W_K(.)(I - I^*)}{-W_K(.)g_I} = \vec{D}_I(y, K, x^*, I; 0_N, g_I) + AE_I,$$

where $AE_I \geq 0$ is the allocative inefficiency measure of dynamic factors. The shadow cost inefficiency measure of dynamic factors is the difference between the shadow value of actual gross investments and the shadow value of optimal gross investments, normalized by the shadow value of the direction vector g_I.

Equation in (13) can be further decomposed in the following way:

(14) $$OE_I = \frac{W_K(.)(I - I^*)}{-W_K(.)g_I} = \sum_{f=1}^{F} \frac{W_{K_f}(.)(I_f - I_f^*)}{-W_K(.)g_I} = \sum_{f=1}^{F} OE_{I_f},$$

where OE_{I_f} is the shadow cost inefficiency measure of the f^{th} dynamic factor. This decomposition allows identifying the dynamic factors that are over-invested ($OE_{I_f} < 0$)

or under-invested ($OE_{I_f} > 0$). The shadow cost inefficiency measures of the F dynamic factors can be all zero or all negative. However, OE_{I_f} cannot be all positive due to property V.4 of V.

The directional variable input distance function is a particular case of the dynamic distance function presented in definition 1 and is given by

(15) $$\vec{D}_x(y,K,x,I;g_x,0_F) = \max\{\beta_x : (x-\beta_x g_x, I) \in V(y\,|\,K)\},$$

with $g_I = 0_F$ and $\vec{D}_x(y,K,x,I;g_x,0_F) \geq 0$. The properties of $\vec{D}_x$ are inherited from the properties of the dynamic directional input distance function in the definition 1. Those properties are similar to the properties of the directional input distance function developed by Chambers, Chung and Färe (1996) including two additional properties: $\vec{D}_x$ is decreasing in I and increasing in K.

Duality between $\vec{D}_x$ and the variable cost function $C(y,K,I,w)$ can also be established. Intuitively, this dual relation can be expressed as the following optimization problems

(16a) $$C(y,K,I,w) = \min_x\{w'x - \vec{D}_x(y,K,x,I;g_x,0_F)w'g_x\},$$

(16b) $$\vec{D}_x(y,K,x,I;g_x,0_F) = \inf_w\left\{\frac{w'x - C(y,K,I,w)}{w'g_x}\right\},$$

$w'g_x \neq 0$.

A variable cost inefficiency measure can be generated from (16a) and (16b) as

(17) $$OE_x = \frac{w'x - C(y,K,I,w)}{w'g_x} = \vec{D}_x(y,K,x,I;g_x,0_F) + AE_x,$$

where $\vec{D}_x(y,K,x,I;g_x,0_F)$ is the technical inefficiency measure of variable inputs and $AE_x \geq 0$ is the allocative inefficiency measure. The cost inefficiency of variable inputs is the normalized difference between actual total variable costs and minimum total variable costs. The normalization is the market value of the direction vector g_x.

The cost inefficiency of variable inputs in (17) can be decomposed as follows:

$$OE_x = \frac{w'x - w'x^*}{w'g_x} = \frac{\sum_{n=1}^{N} w_n (x_n - x_n^*)}{w'g_x} = \sum_{n=1}^{N} OE_{x_n}, \tag{18}$$

where $C(y,K,I,w) = w'x^*$ and OE_{x_n} is the cost inefficiency measure of the n^{th} variable input. The decomposition in (18) allows identifying variable inputs that are either overused ($OE_{x_n} > 0$) or underused ($OE_{x_n} < 0$). The cost inefficiency measures of the *N* inputs can be all zero or all positive. However, OE_{x_n} cannot be all negative due to property V.3 of *V*.

Empirical Models

The empirical implementation of the inefficiency measures is illustrated using DEA models. Dynamic or intertemporal versions of DEA have been developed recently (e.g., Färe and Grosskopf 1996; Nemoto and Goto 1999, 2003; Silva and Stefanou 2003, 2007; Ouellette and Yan 2008). The dynamic DEA models formulated in Färe and Grosskopf (1996) are built on the notions of intermediate outputs and storable inputs. The time interdependence of the production decisions result from the fact that some outputs from an earlier period are used as inputs in the next period and some inputs are storable for one period reducing the input use in this period and increase the input use in the next one. Nemoto and Goto (1999, 2003), Silva and Stefanou (2003, 2007) and Ouellette and Yan (2008) develop dynamic DEA models in the light of the adjustment-cost theory of the firm.

The dynamic DEA models in Nemoto and Goto (1999, 2003) are constructed on the basis of a production possibility set defined in terms of variable inputs, quasi-fixed factors and outputs, where stocks of the quasi-fixed factors at the end of each period are treated as outputs while the stocks of these factors at the beginning of each period as treated as inputs. The dynamic factors (i.e., the change in the level of the quasi-fixed factors) are not explicitly modelled in the firm's production technology. In fact, Ouellette and Yan (2008), pp. 244-245, discuss some limitations of Nemoto and Goto's DEA models. In the dynamic DEA models constructed in Silva and Stefanou (2003, 2007) and Ouellette and Yan (2008), the dynamic factors are explicitly incorporated in the firm's production technology. Silva and Stefanou (2007) consider that investment decisions are irreversible and develop hyperbolic dynamic efficiency measures in the long- and short-run; Ouellette and Yan (2008) consider the possibility of investment and disinvestment and focus on the efficiency of variable inputs.

The dynamic DEA models used in this paper are similar to the DEA models constructed in Silva and Stefanou (2003, 2007). Consider a data series $\left\{\left(y^j, x^j, I^j, K^j, w^j, c^j\right); j=1,\ldots,J\right\}$ representing the observed behavior of each firm *j* at each time *t* and including information on *w* and *c* for each observation *j* at each time *t*. The dynamic directional input distance function measure of technical inefficiency for all factors of production can be generated for each observation *i* as follows:

$$\vec{D}(y^i, K^i, x^i, I^i; g_x, g_I) = \max_{\beta^i, \gamma^j} \beta^i$$

(19)

$$\begin{aligned}
&s.t \\
&y_m^i \leq \sum_{j=1}^{J} \gamma^j y_m^j, \quad m=1,\ldots,M; \\
&\sum_{j=1}^{J} \gamma^j x_n^j \leq x_n^i - \beta^i g_{x_n}, \ n=1,\ldots,N; \\
&I_f^i + \beta^i g_{I_f} - \delta_f K_f^i \leq \sum_{j=1}^{J} \gamma^j (I_f^j - \delta_f K_f^j), \ f=1,\ldots,F; \\
&\gamma^j \geq 0, \ j=1,\ldots,J.
\end{aligned}$$

where γ is the $(J\times 1)$ intensity vector, J is the total number of firms in the sample. The direction vector adopted in the empirical application is $(g_x, g_I) = (x, I)$, i.e. the actual quantities of variable inputs and investments. Note that the output constraints and the variable inputs constraints in (19) reflect, respectively, properties V.7 and V.3 of the adjustment-cost input requirement set. Properties V.4 and V.6 are reflected in the net investment constraints. Due to the "curse of dimensionality" underlying the DEA, the investment constraints are defined in terms of net investment rather than gross investment.[3] The variable input constraints and the investment constraints in (19) imply the adjustment-cost input requirement set is convex.

Note that the inefficiency measures for all factors of production in (11) depend on observed variables $(y^i, K^i, x^i, I^i, w^i, c^i)$ and on the underlying shadow value of capital. The shadow value of capital is an endogenous variable, thus it must be "estimated" simultaneously with the current value of the optimal value function.

The flow version of the current value of the optimal value function for each observation can be generated as

$$
\begin{aligned}
& rW(y^i, K^i, w^i, c^i) = \min_{x, I, \gamma} \left[w^{i\prime} x + c^{i\prime} K^i + W_K^i (I - \delta K^i) \right] \\
& \quad s.t \\
& \quad \sum_{j=1}^{J} \gamma^j y_m^j \geq y_m^i, \quad m = 1,\ldots,M; \\
(20) \quad & \quad x_n \geq \sum_{j=1}^{J} \gamma^j x_n^j, \quad n = 1,\ldots,N; \\
& \quad \sum_{j=1}^{J} \gamma^j (I_f^j - \delta_f K_f^j) \geq I_f - \delta_f K_f^i, \quad f = 1,\ldots,F; \\
& \quad \gamma^j \geq 0, \ j = 1,\ldots,J; \\
& \quad x_n \geq 0, \ n = 1,\ldots,N; \\
& \quad I_f \geq 0, \ f = 1,\ldots,F;
\end{aligned}
$$

[3] In the dynamic DEA model developed by Silva and Stefanou (2007), the investment constraints are defined in terms of gross investment.

where $W_K^i = W_K(y^i, K^i, w^i, c^i)$ is the vector of the shadow value of capital for observation *i*, *i=1,…,J*. The Kuhn-Tucker conditions of (20) are

$$
\begin{aligned}
&w_n^i - \mu_n^x \geq 0, x_n^* \geq 0, x_n^*(w_n^i - \mu_n^x) = 0, \; n = 1,..., N; \\
&-\sum_{m=1}^{M} \mu_m^y y_m^j + \sum_{n=1}^{N} \mu_n^x x_n^j - \sum_{f=1}^{F} \mu_f^I (I_f^j - \delta_f K_f^j) \geq 0, \\
&\gamma^{j*} \geq 0, \; \gamma^{j*}[.] = 0, \; j = 1,..., J; \\
(21) \qquad &\sum_{j=1}^{J} \gamma^{j*} y_m^j - y_m^i \geq 0, \; \mu_m^y \geq 0, \; \mu_m^y (\sum_{j=1}^{J} \gamma^{j*} y_m^j - y_m^i) = 0, \; m = 1,..., M; \\
&x_n^* - \sum_{j=1}^{J} \gamma^{j*} x_n^j \geq 0, \; \mu_n^x \geq 0, \; \mu_n^x (x_n^* - \sum_{j=1}^{J} \gamma^{j*} x_n^j) = 0, \; n = 1,..., N; \\
&W_{K_f}^i + \mu_f^I \geq 0, \; I_f^* \geq 0, \; I_f^* (W_{K_f}^i + \mu_f^I) = 0, \; f = 1,..., F; \\
&\sum_{j=1}^{J} \gamma^{j*} (I_f^j - \delta_f K_f^j) - (I_f - \delta_f K_f^i) \geq 0, \; \mu_f^I \geq 0, \; \mu_f^I [...] = 0, \; f = 1,..., F;
\end{aligned}
$$

where the dual variables μ_m^y and μ_n^x are the current value of the Lagrangian multipliers associated with the constraint on the output *m* and the variable input *n*, respectively. The dual variable μ_f^I is the current value of the Lagrangian multiplier associated with the constraint on net investment of the quasi-fixed factor *f*. For an interior solution, the negative value of the shadow value of capital $(-W_{K_f}^i)$ is equal to μ_f^I, *f =1,…,F*. This dual variable can be interpreted as the marginal cost of adjustment for the quasi-fixed factor *f*.

The problem in (20) can be solved using the Linear Complementarity Problem (LCP) as in Silva and Stefanou (2007) by expressing the Kuhn-Tucker conditions in (21) in a LCP form, given that $-W_{K_f}^i = \mu_f^I$, *f* =1,…,*F*. Alternatively, the flow version of the current value of the optimal value function for each observation can be generated using the dual of problem (20):

$$
(22) \quad \begin{aligned}
&\max_{\mu_m^y,\mu_n^x,\mu_f^I} \{\sum_{m=1}^{M} \mu_m^y y_m^i + \sum_{f=1}^{F} c_f^i K_f^i\} \\
&s.t. \\
&\quad \mu_n^x - w_n^i \le 0,\ n = 1,...,N; \\
&\quad \sum_{m=1}^{M} \mu_m^y y_m^j + \sum_{f=1}^{F} \mu_f^I (I_f^j - \delta_f K_f^j) - \sum_{n=1}^{N} \mu_n^x x_n^j \le 0,\ j = 1,...,J.
\end{aligned}
$$

The solution obtained by solving (20) or (22) provides the optimal variable input and dynamic factor vectors, the flow version of current value of the optimal value function and the value of the underlying shadow values of the quasi-fixed factors. Using these values, the dynamic cost inefficiency measure in (11) can be generated. Given the solution of problem (19) and the dynamic cost inefficiency measure, the allocative inefficiency measure of all factors of production in (11) is calculated residually.

The technical inefficiency measure for dynamic factors in (13) can be generated for each observation as follows:

$$
(23) \quad \begin{aligned}
&\vec{D}_I (y^i, K^i, x^{i*}, I^i; 0_N, g_I) = \max_{\beta_I^i,\gamma^j} \beta_I^i \\
&\quad s.t \\
&\quad y_m^i \le \sum_{j=1}^{J} \gamma^j y_m^j,\quad m = 1,...,M; \\
&\quad \sum_{j=1}^{J} \gamma^j x_n^j \le x_n^{i*},\quad n = 1,...,N; \\
&\quad I_f^i + \beta_I^i g_{I_f} - \delta_f K_f^i \le \sum_{j=1}^{J} \gamma^j (I_f^j - \delta_f K_f^j) \ge,\ \ f = 1,...,F; \\
&\quad \gamma^j \ge 0,\ j = 1,...,J.
\end{aligned}
$$

The direction vector is defined as $(g_x, g_I) = (0_N, I)$.

Given the technical inefficiency measure for the dynamic factors in (23), the shadow value of capital and the optimal level of the dynamic factors from solving (20) or (22), the allocative inefficiency measure can be calculated residually using (13). The shadow cost inefficiency of each dynamic factor can be computed using (14).

The technical inefficiency measure of variable inputs in (15) for observation i is generated as follows:

$$\vec{D}_x(y^i, K^i, x^i, I^{i*}; g_x, 0_F) = \max_{\beta_x^i, \gamma^j} \beta_x^i$$

$$\text{(24)} \qquad \begin{aligned} & s.t \\ & y_m^i \le \sum_{j=1}^{J} \gamma^j y_m^j, \quad m = 1, \ldots, M; \\ & \sum_{j=1}^{J} \gamma^j x_n^j \le x_n^i - \beta_x^i g_{x_n}, \quad n = 1, \ldots, N; \\ & I_f^{i*} - \delta_f K_f^i \le \sum_{j=1}^{J} \gamma^j (I_f^j - \delta_f K_f^j) \ge, \quad f = 1, \ldots, F; \\ & \gamma^j \ge 0, \ j = 1, \ldots, J, \end{aligned}$$

The optimal level of dynamic factors for each observation is obtained from solving (20) or (22). The direction vector adopted is $(g_x, g_I) = (x, 0_F)$.

The minimum variable cost for each firm can be generated as

$$C(y^i, K^i, I^{i*}, w^i) = \min_{x, \gamma} w^{i\prime} x$$

$$\text{(25)} \qquad \begin{aligned} & s.t \\ & \sum_{j=1}^{J} \gamma^j y_m^j \ge y_m^i, \quad m = 1, \ldots, M; \\ & x_n \ge \sum_{j=1}^{J} \gamma^j x_n^j, \quad n = 1, \ldots, N; \\ & \sum_{j=1}^{J} \gamma^j (I_f^j - \delta_f K_f^j) \ge I_f^{i*} - \delta_f K_f^i, \quad f = 1, \ldots, F; \\ & \gamma^j \ge 0, \ j = 1, \ldots, J \end{aligned}$$

Given $C(.)$ in (25), the allocative inefficiency of variable inputs is determined residually for each firm using (17). The cost inefficiency of each variable input can be calculated using (18).

Empirical Results

In this section, the input-based dynamic efficiency measures are illustrated on panel data of Dutch glasshouse firms over the period 1997-1999. Data on specialised vegetables firms covering the period 1997-1999 are obtained from a stratified sample of Dutch glasshouse firms keeping accounts on behalf of the LEI accounting system. The data contain 265 observations on 103 firms, so the panel is unbalanced.

One output and six inputs (energy, materials, services, structures, installations and labour) are distinguished. Output mainly consists of vegetables, potted plants, fruits and flowers. Energy consists of gas, oil and electricity, as well as heat deliveries by electricity plants. Materials consist of seeds and planting materials, pesticides, fertilisers and other materials. Services are those provided by contract workers and from storage and delivery of outputs.

Quasi-fixed inputs are structures (buildings, glasshouses, land and paving) and installations. Capital in structures and installations is measured at constant 1991 prices and is valued in replacement costs.[4] Labour is a fixed input and is measured in quality-corrected man-years, including family as well as hired labour. Labour is assumed to be a fixed input because a large share of total labour consists of family labour. Flexibility of hired labour is further restricted by the presence of permanent contracts and by the fact that hiring additional labour involves search costs for the firm operator. The quality correction of labour is performed by the LEI and is necessary to aggregate labour from able-bodied adults with labour supplied by young people (e.g., young family members) or partly disabled workers.

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[4] The deflators for structures and installations are calculated from the data supplied by the LEI accounting system. Comparison of the balance value in year t and the balance value in year t-1 gives the yearly price correction used by the LEI. This price correction is used to construct a price index for structures and a price index for installations. These price indices are used as deflators.

Tornqvist price indexes are calculated for output, variable inputs and quasi-fixed inputs with prices obtained from the LEI/CBS. The price indexes vary over the years but not over the firms, implying differences in the composition of inputs and output or quality differences are reflected in the quantity (Cox and Wohlgenant 1986). Implicit quantity indexes are generated as the ratio of value to the price index. A more detailed description of the data can be found in Table 1.

Inefficiency scores are generated for each horticulture firm in each year over the 1997-1999 period.[5] Table 2 reports average values of technical, allocative and cost inefficiency of variable and dynamic factors of production for each year and for the whole time period.

Table 2 shows that the average cost inefficiency over the 1997-1999 period is 0.44 implying that substantial cost savings can be obtained. Technical inefficiency is the largest component of cost inefficiency for each year and for the whole time period, ranging between 0.39 (1997) and 0.26 (1999). The average allocative inefficiency of 0.10 suggests that Dutch vegetables firms can reduce costs through a better mix of variable and dynamic factors in the light of prevailing prices.

Table 3 presents cost inefficiency for each variable input using (18). The results in Table 3 suggest there is, on average, overuse of all variable inputs in the whole period 1997-1999. Overuse for energy and materials is particularly high as cost inefficiency for energy and materials is, on average, 0.32 and 0.26, respectively. Cost inefficiency is lowest for services; for 1999 there is a small underuse rather than overuse of this variable input. The relatively large cost inefficiency for energy may be due to the fact that firms use a large variety of heating technologies. A group of firms uses more advanced and efficient technologies such as co-generators, heat storage and heat

[5] Due to space limitations, inefficiency levels are not reported for each firm. The inefficiency scores by firm are available from the authors upon request.

delivery by electricity plants, whereas a majority of firms still uses traditional heating technologies based on a combustion heater (Oude Lansink and Silva 2003).

Table 4 presents cost inefficiency for individual dynamic factors of production for individual years and for the whole time period 1997-1999. The results suggest that firms neither over-invest nor under-invest in structures and installations in 1997 and 1998. For the year 1999, the values of -0.40 and -0.02 indicate a large overinvestment in structures and a small overinvestment in installations. Inspection of the data reveals that average investments in structures and installations are indeed much higher in 1999 than in the two preceding years.[6] Therefore, firms have substantially increased their investment level in 1999 compared to the previous years. However, they have been over-investing in both structures and installations.

Concluding Remarks

In this paper, a dynamic directional input distance function is defined and characterized in the context of the adjustment-cost model of the firm and shown to be a complete function representation of the adjustment-cost production technology. Intertemporal duality is established between the dynamic directional input distance function and the current value of the optimal value function of the intertemporal cost minimization problem.

The dynamic directional input distance function is a representation of the adjustment-cost technology that provides dynamic difference measures of relative efficiency, as opposed to ratio measures (e.g., Nemoto and Goto 2003) or hyperbolic measures as in Silva and Stefanou (2007). A method for measuring dynamic input-based efficiency is developed by exploiting the intertemporal duality. Furthermore, this

[6] Investments in structures in 1997, 1998 and 1999 are 38, 39 and 66 (1000 guilders), respectively. Investments in installations in these years are 38, 31 and 57 (1000 guilders).

paper shows that the dynamic inefficiency measures easily disentangle the contribution of individual variable and dynamic factors of production to inefficiency.

The dynamic input inefficiency measures are applied to panel data of Dutch glasshouse firms over the period 1997-1999. The results suggest that these firms can achieve substantial cost savings, particularly by improving technical efficiency of the variable and dynamic factors of production. Analysis of the contribution of individual variable factors to inefficiency shows that energy and material are among the least efficiently employed. Further analysis of the dynamic factors reveals that firms over-invested in structures in the year 1999.

There are a number of directions future research can move. In a dynamic production context, the impact of technological progress and uncertainty cannot be neglected. Specifically, the effects of (price and production) uncertainty and risk preferences on economic decisions are likely to be significant with consequences on the level of efficiency achieved by decision-makers.

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Table 1: Variables and Descriptive Statistics

Variable	Dimension	Mean	Standard Deviation
Quantities			
Output	1000 Guilders	1124.20	984.60
Energy	1000 Guilders	132.79	121.99
Materials	1000 Guilders	124.82	99.91
Services	1000 Guilders	85.59	73.56
Structures	1000 Guilders	833.38	697.18
Installations	1000 Guilders	229.39	243.31
Labor	Man years	6.62	5.17
Investments Structures	1000 Guilders	46.70	156.43
Investments Installations	1000 Guilders	41.34	128.90
Prices			
Energy	1991=1	1.14	0.03
Materials	1991=1	1.05	0.02
Services	1991=1	0.94	0.02
Structures	1991=1	1.51	0.13
Installations	1991=1	1.08	0.03

Table 2: Technical, Allocative and Cost Inefficiency of All Factors of Production

Period	TE	AE	OE
1997	0.39	0.09	0.48
1998	0.34	0.11	0.45
1999	0.26	0.13	0.39
1997-1999	0.33	0.10	0.44

Table 3 Cost Inefficiency of Variable Factors of Production

Period	Energy	Materials	Services
1997	0.33	0.26	0.07
1998	0.33	0.25	0.03
1999	0.31	0.27	-0.02
1997-1999	0.32	0.26	0.03

Table 4 Cost Inefficiency of Dynamic Factors of Production

Period	Structures	Installations
1997	0.01	0.01
1998	-0.00	0.00
1999	-0.40	-0.02
1997-1999	-0.12	-0.00

Appendix

Proof of Lemma 1: The time index t is omitted for the sake of a clear exposition.

D.1 Assume $(x,I)\in V(y\,|\,K)$ and $(x',I')\in V(y\,|\,K)$ and let $x''=\alpha x+(1-\alpha)x'$ and $I''=\alpha I+(1-\alpha)I'$, $\alpha\in(0,1)$. By the strict convexity of $V(y|K)$, $(x'',I'')\in \operatorname{int}V(y\,|\,K)$. By the definition 1,

$$\left[x-\vec{D}(y,K,x,I;g_x,g_I)g_x, I+\vec{D}(y,K,x,I;g_x,g_I)g_I\right]\in V(y\,|\,K),$$

$$\left[x'-\vec{D}(y,K,x',I';g_x,g_I)g_x, I'+\vec{D}(y,K,x',I';g_x,g_I)g_I\right]\in V(y\,|\,K),$$

$$\left[x''-\vec{D}(y,K,x'',I'';g_x,g_I)g_x, I''+\vec{D}(y,K,x'',I'';g_x,g_I)g_I\right]\in V(y\,|\,K).$$

Let $\vec{D}''=\alpha\vec{D}(y,K,x,I;g_x,g_I)+(1-\alpha)\vec{D}(y,K,x',I';g_x,g_I)$. By the strict convexity of $V(y|K)$, $(x''-\vec{D}''g_x, I''+\vec{D}''g_I)\in \operatorname{int}V(y\,|\,K)$. By the definition 1, this means that

$\vec{D}(y,K,x'',I'';g_x,g_I)>\vec{D}''=\alpha\vec{D}(y,K,x,I;g_g,g_I)+(1-\alpha)\vec{D}(y,K,x',I';g_g,g_I)$.

D.2 This property follows directly from the definition 1.

D.3 Since $y'\geq y$, then, by property V.7, $V(y'\,|\,K)\subset V(y\,|\,K)$. Thus, by the definition 1, it must be the case that $\vec{D}(y',K,x,I;g_x,g_I)<\vec{D}(y,K,x,I;g_x,g_I)$.

D.4 Assume property V.3 holds and let $x'\geq x$. By the definition of $\vec{D}$,

$$\left[x-\vec{D}(y,K,x,I;g_x,g_I)g_x, I+\vec{D}(y,K,x,I;g_x,g_I)g_I\right]\in V(y\,|\,K),$$

$$\left[x'-\vec{D}(y,K,x',I;g_x,g_I)g_x, I+\vec{D}(y,K,x',I;g_x,g_I)g_I\right]\in V(y\,|\,K).$$

It must be the case that

$$x'-\vec{D}(y,K,x',I;g_x,g_I)g_x\leq x'-\vec{D}(y,K,x,I;g_x,g_I)g_x.$$

Then, $\vec{D}(y,K,x',I;g_x,g_I)>\vec{D}(y,K,x,I;g_x,g_I)$.

D.5 The proof is similar to the previous one.

D.6 The proof of this property is similar to the proof of property D.3.

D.7 The proof follows directly from the definition 1.

D.8 This property follows from property V.1 and V.2.

Proof of Equation (6):

The Lagrangian problem associated with the H-J-B equation in (5) is

$$
\begin{aligned}
& rW(y,K,w,c) = rW(y,K,w,c;0) \\
\text{(A.1)} \qquad & = \min_{x,I} \{ w'x + c'K + W_K(y,K,w,c)(I-\delta K) \\
& \qquad + \lambda(\vec{D}(y,K,x,I;g_x,g_I) - 0) \}.
\end{aligned}
$$

Consider the following H-J-B equation

$$
\text{(A.2)} \qquad rW(y,K,w,c;\alpha) = \min_{x,I} \{ w'x + c'K + W_K(y,K,w,c;\alpha)(I-\delta K) : \vec{D}(y,K,x,I;g_x,g_I) \geq \alpha \},
$$

where the Lagrangian problem associated with (A.2) is

$$
\text{(A.3)} \qquad rW(y,K,w,c;\alpha) = \min_{x,I} \{ w'x + c'K + W_K(y,K,w,c;\alpha)(I-\delta K) + \lambda(\vec{D}(y,K,x,I;g_x,g_I) - \alpha) \}.
$$

Applying the prototype envelope theorem to (A.3) yields

$$
\text{(A.4)} \qquad r\frac{\partial W(y,K,w,c;\alpha)}{\partial \alpha} = W_{K\alpha}(.)(I-\delta K) - \lambda .
$$

Given property D.2, the Lagrangian problem in (A.3) can be rewritten as

$$
\text{(A.5)} \qquad rW(y,K,w,c;\alpha) = \min_{x,I} \{ w'x + c'K + W_K(y,K,w,c;\alpha)(I-\delta K) + \lambda(\vec{D}(y,K,x-\alpha g_x, I+\alpha g_I; g_x,g_I) - 0) \},
$$

or, equivalently

$$
\begin{aligned}
& rW(y,K,w,c;\alpha) = \min_{x,I} \{ w'(x-\alpha g_x) + c'K + W_K(y,K,w,c;\alpha)(I+\alpha g_I - \delta K) \\
\text{(A.6)} \qquad & \qquad + \lambda(\vec{D}(y,K,x-\alpha g_x, I+\alpha g_I; g_x,g_I) - 0) \} \\
& \qquad + \alpha(w'g_x - W_K(y,K,w,c;\alpha)g_I).
\end{aligned}
$$

Applying the prototype envelope theorem to (A.6) yields

(A.7) $$r\frac{\partial W(y,K,w,c;\alpha)}{\partial\alpha} = W_{K\alpha}(.)(I-\delta K) + w' g_x - W_K(.)g_I\,.$$

From (A.4) and (A.7), one can establish that

(A.8) $$\lambda = W_K(.)g_I - w' g_x\,.$$

Since $rW(y,K,w,c) = rW(y,K,w,c;0)$, then

$$\begin{aligned} rW(y,K,w,c) = \min_{x,I}\{ & w'x + c'K + W_K(y,K,w,c)(I-\delta K) \\ & + (W_K(y,K,w,c)g_I - w' g_x)\vec{D}(y,K,x,I;g_x,g_I)\}. \end{aligned}$$

Proof of equation (7):

A procedure similar to the one employed by Kamien and Schwartz (1991), section 4, pp. 136-41, in the context of intertemporal profit maximization, is used to prove equation (7).

Consider the H-J-B equation in (6). Differentiating (6) with respect to K_f and using the static envelop theorem yields

(A.9) $$\sum_{h=1}^{F} W_{K_h K_f}(.)\ (I_h^* - \delta_h K_h) - W_{K_f}(.)\ (r+\delta_f) = -c_f - \lambda\vec{D}_{K_f}(.)\,.$$

Totally differentiating $W_{K_f}(y,K,w,c)$ yields

(A.10) $$\dot{W}_{K_f}(.) = \sum_{h=1}^{F} W_{K_f K_h}(.)\dot{K}_h\,,$$

assuming that $\dot{y} = 0_M$, $\dot{w} = 0_N$, and $\dot{c} = 0_F$.

Substituting (A.10) in (A.9), yields

(A.11) $$\dot{W}_{K_f}(.) - W_{K_f}(.)(r+\delta_f) = -c_f - \lambda\vec{D}_{K_f}(.)\,.$$

The differential equation in (A.11) can be used to show the essence of the current shadow value of a unit of the quasi-fixed factor *f*. Multiply both sides of (A.11) by the integrating factor $e^{-(r+\delta_f)t}$, and integrate from *t* to +∞

(A.12) $$\int_t^{+\infty} e^{-(r+\delta_f)s}\left[\dot{W}_{K_f}(.) - W_{K_f}(.)(r+\delta_f)\right]ds = \int_t^{+\infty} e^{-(r+\delta_f)s}\left(-c_f - \lambda \vec{D}_{K_f}(.)\right)ds\,.$$

Calculating the integral on the left hand-side in (A.12), yields

(A.13) $$W_{K_f}(.)e^{-(r+\delta_f)t} = \int_t^{+\infty} e^{-(r+\delta_f)s}\left(-c_f - \lambda \vec{D}_{K_f}(.)\right)ds\,.$$

The previous equation can be rewritten as follows

(A.14) $$W_{K_f}(.) = \int_t^{+\infty} e^{-r(s-t)}\left(-e^{-\delta_f(s-t)}c_f - e^{-\delta_f(s-t)}\lambda \vec{D}_{K_f}(.)\right)ds\,.$$

Proof of property W.3 in theorem 1 shows that $\lambda < 0$ and $W_{K_f} < 0$. By property D.6, $\vec{D}_{K_f} > 0$. Equation (A.14) implies that the marginal value of a unit of the quasi-fixed factor *f*, at time *t*, is the discounted stream of the net marginal benefits, $\left(-e^{-\delta_f(s-t)}c_f - e^{-\delta_f(s-t)}\lambda \vec{D}_{K_f}(.)\right)$, it generates from *t* to infinity. The value of a marginal unit of the quasi-fixed factor *f* reflects its depreciation rate δ_f. At time *s*, *s* > *t*, the contribution of a unit of the quasi-fixed factor *f* is only a fraction $e^{-\delta_f s}$ of its contribution at time *t*.

Proof of Theorem 1:

W.1 Given assumption (a.1), *W* is a real-valued function. By the dynamic envelope theorem and the principle of optimality (e.g., Caputo 2005, chapter nine, pp. 231-242), $W_K(y,K,w,c) = \theta^*(y,K,w,c)$. Given assumption (a.1), $W_K(.) \in C^{(2)}$. Now, it is left to show that the second-order partial derivatives of *W* with respect to (*y*,*w*,*c*) are continuous. Applying the static envelope theorem to problem (6),

$$rW_{w_n}(y,K,w,c) = x_n^*(y,K,w,c) + \sum_{f=1}^{F} W_{K_f w_n}(y,K,w,c)\left[I_f^*(y,K,w,c) - \delta_f K_f\right], n = 1,...,N,$$

$$rW_{c_f}(y,K,w,c) = K_f + \sum_{l=1}^{F} W_{K_l c_f}(y,K,w,c)\left[I_l^*(y,K,w,c) - \delta_l K_l\right], \; f = 1,...,F,$$

$$rW_{y_m}(y,K,w,c) = \lambda^*(.)\vec{D}_y(.) + \sum_{f=1}^{F} W_{K_f y_m}(y,K,w,c)\left[I_f^*(y,K,w,c) - \delta_f K_f\right], \; m = 1,...,M.$$

Given property D.9 and assumption (a.1), the right-hand sides of the previous equations are $C^{(1)}$ functions of (*y*,*w*,*c*). Thus, the second-order partial derivatives of *W* with respect to output levels and prices (*y*,*w*,*c*) are continuous functions, implying $W(y,K,w,c) \in C^{(2)}$.

W.2 This property follows considering the intertemporal problem in (3) and property V.7.

W.3 Consider the H-J-B equation in (6). By the optimality conditions for an interior solution

$$w_n + \lambda^* \vec{D}_{x_n} = 0, \; n = 1,...,N, \quad \text{and} \qquad W_{K_f} + \lambda^* \vec{D}_{I_f} = 0, \; f = 1,...,F \; .$$

From properties D.4 and D.5 and these optimality conditions, it must be the case that $W_K < 0^F$.

W.4 In the proof of property W.1, the following equations were established

$$rW_w(.) = x^* + W_{Kw}(.)'(I^* - \delta K), \;\; rW_c(.) = K + W_{Kc}(.)'(I^* - \delta K) \;,$$

and $rW_y(.) = \lambda^* \vec{D}_y(.) + W_{Ky}(.)'(I^* - \delta K)$.

The first two equations establish properties (c) and (d) in W.4. From the last equation, it can be established property W.4 (a) using properties W.3 and D.3.

Proceeding in a similar way, it can be established

$$(r+\delta)W_K(.) - c - W_{KK}(.)'(I^* - \delta K) = \lambda^* \vec{D}_K(y,K,x^*,I^*; g_x, g_I)$$

Using properties D.6 and W.3, property W.4 (b) is established.

W.5 Let the triplet $(x(s), I(s), K(s))$ be optimal for (w,c) and $K(t) = K_t$.

$$W(y, K_t, w, c) = \int_t^{\infty} e^{-r(s-t)} \left[w'x(s) + c'K(s)\right] ds$$
$$\leq \int_t^{\infty} e^{-r(s-t)} \left[w'x^o(s) + c'K^o(s)\right] ds, \ \forall (x^o(s), I^o(s)) \in V(y(s) : K^o(s))$$

Also,

$$\int_t^{\infty} e^{-r(s-t)} \left[\alpha w'x(s) + \alpha c'K(s)\right] ds \leq \int_t^{\infty} e^{-r(s-t)} \left[\alpha w'x^o(s) + \alpha c'K^o(s)\right] ds, \ \forall \alpha > 0 .$$

This is equivalent to state

$$W(y, K_t, \alpha w, \alpha c) = \int_t^{\infty} e^{-r(s-t)} \left[\alpha w'x(s) + \alpha c'K(s)\right] ds = \alpha W(y, K_t, w, c) .$$

W.6 (a) Let $w^1 \geq w^2$ and let the triplet $(x^i(s), I^i(s), K^i(s))$ be optimal for $w = w^i$ and $K^i(t) = K_t$, i = 1,2.

$$W(y, K_t, w^1, c) = \int_t^{\infty} e^{-r(s-t)} \left[w^{1\prime} x^1(s) + c'K^1(s)\right] ds$$
$$> \int_t^{\infty} e^{-r(s-t)} \left[w^{2\prime} x^1(s) + c'K^1(s)\right] ds$$
$$\geq \int_t^{\infty} e^{-r(s-t)} \left[w^{2\prime} x^2(s) + c'K^2(s)\right] ds$$
$$= W(y, K_t, w^2, c)$$

The first equality results from the definition of the optimal value function W in problem (4). The second inequality results from the fact that $w^1 \geq w^2$ and the optimal triplet is interior; and the third inequality is a consequence of optimality.

(b) It can be proved following a similar procedure as in (a).

W.7 Let (w^i, c^i), $i = 1,2$, and $(w^{\alpha}, c^{\alpha}) = \alpha(w^1, c^1) + (1-\alpha)(w^2, c^2)$, $\alpha \in [0,1]$. Let the triplet $(x^j(s), I^j(s), K^j(s))$ be optimal for (w^j, c^j), $j = 1,2,\alpha$, and $K^j(t) = K_t$, $\forall j$.

$$W(y,K_t,w^\alpha,c^\alpha)=\int_t^\infty e^{-r(s-t)}\left[w^{\alpha'}x^\alpha(s)+c^{\alpha'}K^\alpha(s)\right]ds$$
$$=\int_t^\infty e^{-r(s-t)}\left\{\alpha\left[w^{1'}x^\alpha(s)+c^{1'}K^\alpha(s)\right]+(1-\alpha)\left[w^{2'}x^\alpha(s)+c^{2'}K^\alpha(s)\right]\right\}ds$$
$$\geq\alpha\int_t^\infty e^{-r(s-t)}\left[w^{1'}x^1(s)+c^{1'}K^1(s)\right]ds+(1-\alpha)\int_t^\infty e^{-r(s-t)}\left[w^{2'}x^2(s)+c^{2'}K^2(s)\right]ds$$
$$=\alpha W(y,K_t,w^1,c^1)+(1-\alpha)W(y,K_t,w^2,c^2)$$

W.8 Given assumption (a.2), $x^0=x^*(y,K,w^0,c^0)$ and $I^0=I^*(y,K,w^0,c^0)$ solve the H-J-B problem in (4) when $(w,c)=(w^0,c^0)$. Then, the primal-dual problem corresponding to problem (4) is as follows

$$\begin{aligned}0=\min_{w,c}\Big\{&w'x^0+c'K+W_K(y,K,w,c)(I^0-\delta K)\\&+(W_K(y,K,w,c)g_I-wg_x)\vec{D}(y,K,x^0,I^0;g_x,g_I)\Big\}\\&-rW(y,K,w,c).\end{aligned}\tag{A.9}$$

The price vector (w^0,c^0) is optimal for problem (A.9) since $x^0=x^*(y,K,w^0,c^0)$ and $I^0=I^*(y,K,w^0,c^0)$. Since the dynamic directional distance function is independent of (w, c), (A.9) can be rewritten as

$$\vec{D}(y,K,x^0,I^0;g_x,g_I)=\min_{w,c}\left\{\frac{w'x^0+c'K+W_K(.)(I^0-\delta K)-rW(y,K,w,c)}{w'g_x-W_K(.)g_I}\right\}.\tag{A.10}$$

Given the arbitrariness of the choice of point (y,K,w^0,c^0), problem in (A.10) is equivalent to problem in W.8. Thus, $F(y,K,x,I;g_x,g_I)=\vec{D}_i(y,K,x,I;g_x,g_I)$. Strict concavity and non-negativity follows directly from property D.1 and equation (2), respectively.

Proof of Theorem 2:

By property W.8 in theorem 1, $F(y,K,x,I;g_x,g_I)$ is well-defined. Since $\vec{D}(.) = F(.)$, then $\vec{D}$ is well-defined.

D.1 and D.8 From property W.8 in theorem 1 follows that $\vec{D}$ is strictly concave in (*x*,*I*), given *y* and *K*. D.8 follows from property W.1.

D.2 and D.7 Both properties follow directly from the definition of *F* in W.8.

D.3 $\vec{D}_y{}' = F_y{}' = \dfrac{W_{Ky}(.)'(I-\delta K) - rW_y(.)'}{w'g_x - W_K(.)g_I} < 0^M$, applying the static envelope theorem to problem in W.8 and using properties W.2, W.3 and W.4(a).

D.4 $\vec{D}_x{}' = F_x{}' = \dfrac{w_n}{w'g_x - W_K(.)g_I} > 0^N$, applying the static envelope theorem to problem in W.8 and using property W.3.

D.5 $\vec{D}_I{}' = F_I{}' = \dfrac{W_K(.)'}{w'g_x - W_K(.)g_I} < 0^F$, applying the static envelope theorem to problem in W.8 and using property W.3.

D.6 $\vec{D}_K{}' = F_K{}' = \dfrac{c + W_{KK}(.)(I-\delta K) - (rI^F + \delta)W_K(.)'}{w'g_x - W_K(.)g_I} > 0^F$, applying the static envelope theorem to problem in W.8 and using properties W.3 and W.4(b).

D.9 Inspection of the properties D.3-D.6, $\vec{D}_l = F_l$, $l = y, K, x, I$, and $F_l(.) \in C^{(1)}$.