Modelling Changes in the Unconditional Variance of Long Stock Return Series

Cristina Amado∗
University of Minho and NIPE
Campus de Gualtar, 4710-057 Braga, Portugal

Timo Teräsvirta†
CREATES, School of Economics and Management, Aarhus University
Building 1322, DK-8000 Aarhus, Denmark

May 2011

Work in Progress. Please do not cite.

∗E-mail: camado@eeg.uminho.pt
†E-mail: tterasvirta@econ.au.dk

Acknowledgements: This research has been supported by the Danish National Research Foundation. The first author would like to acknowledge financial support from the Louis Fraenckels Stipendiefond. Part of this research was done while the first author was visiting CREATES, Aarhus University, whose kind hospitality is gratefully acknowledged. The responsibility for any errors and shortcomings in this paper remains ours.
Abstract

In this paper we develop a testing and modelling procedure for describing the long-term volatility movements over very long return series. For the purpose, we assume that volatility is multiplicatively decomposed into a conditional and an unconditional component as in Amado and Teräsvirta (2011). The latter component is modelled by incorporating smooth changes so that the unconditional variance is allowed to evolve slowly over time. Statistical inference is used for specifying the parameterization of the time-varying component by applying a sequence of Lagrange multiplier tests. The model building procedure is illustrated with an application to the daily returns of the DJIA index covering a period of eighty three years of financial market history. Two major conclusions are as follows. First, the LM tests strongly reject the assumption of constancy of the unconditional variance. Second, the results show that the long-memory property in volatility may be explained by ignored changes in the unconditional variance of the long series.

*JEL classification:* C12; C22; C51; C52

*Key words:* Model specification; Conditional heteroskedasticity; Lagrange multiplier test; Time-varying unconditional variance; Long financial time series; Volatility persistence.
1 Introduction

The observation that deterministic shifts in long return series can generate long-memory behaviour has received much attention in recent years. Most of the work in this topic is related with the study of the behaviour of standard statistical tools and model misspecification under nonstationarity. Early studies include Diebold (1986) and Lamoureux and Lastrapes (1990) who suggested that occasional level shifts in the intercept of the first-order GARCH model can bias the estimation towards an integrated GARCH model. More recently, Mikosch and Stărică (2004) argued that the so-called ‘integrated GARCH effect’ is caused by the nonstationary behaviour of very long return series. They show how the long-range dependence in volatility and the IGARCH effect may be explained by neglected deterministic changes in the unconditional variance of the stochastic process. Moreover, Granger and Hyung (2004) claimed that occasional breaks in a long time series of absolute stock returns can also explain the observed slow decay of the autocorrelation functions of absolute returns in long return series.

It is well documented that shocks to the conditional variance of the standard GARCH model of Bollerslev (1986) decay at an exponential rate. This has motivated the development of more flexible models to describe the observed dependence structure in financial market volatility. One of these models is the Fractionally Integrated GARCH model of Baillie, Bollerslev, and Mikkelsen (1996) which belongs to the class of long-memory models. In these processes, shocks to the conditional variance decay at a slow hyperbolic rate which is more strongly supported by financial data than the GARCH model. A generalization of the FIGARCH model was recently proposed by Baillie and Morana (2007) in which they allow the intercept to change deterministically according to the flexible functional form of Gallant (1984).

The question of explicitly modelling nonstationarity in stock market volatility has, however, received somewhat less attention. There have been some attempts to incorporate nonstationarity directly into the model. Stărică and Granger (2005) introduced a nonstationary approach in which the returns are modelled as nonstationary sequence of independent random variables with time-varying unconditional variance but their model does not allow for volatility clustering. More recently, Engle and Gonzalo Rangel (2008) proposed modelling the volatility process by a multiplicative decomposition into a nonstationary and a stationary component. The nonstationary component (or the unconditional variance) is described by an exponential spline, and the stationary component (or the short-run dynamics of volatility) follows a first-order GARCH process.

This paper addresses the issue of modelling deterministic changes in the unconditional variance of long return series. It is assumed that volatility is modelled by decomposing the variance into a conditional and an unconditional component as in Amado and Teräsvirta (2011). The conditional variance follows a GARCH process, and describes the short-run dynamics of volatility. The nonstationary component of volatility describes the long-volatility dynamics, and it is represented by a linear combination of logistic transition functions. Statistical inference is used for specifying the parametric structure of the time-varying component by applying a sequence of Lagrange multiplier tests. Our modelling strategy is applied to describe the long-run properties of the long daily Dow Jones Industrial Average (DJIA) return series from 1920 to 2003. One may expect that the longer the observation period, the more likely the occurrence of structural changes or shifts in the second unconditional moment of returns. The test results strongly support the time-variation of the unconditional variance in the period under study. The estimation results indicate that the strongest deterministic changes in the unconditional variance are associated with the largest economic recessions. This in turn suggests that the unconditional variance behaviour may be related to the evolution of the deterministic conditions in the economy. Our findings also suggest that the observed long-memory property in volatility may well be due to deterministic changes in the unconditional variance of the return series.
The paper is organized as follows. The TV-GARCH model and the modelling strategy are presented in Section 2. Details regarding the estimation of the model are discussed in Section 3. Section 4 contains the application. In Section 5 we show by a small Monte Carlo simulation how ignored deterministic changes in the unconditional variance affect the estimation of a misspecified model. Finally, Section 6 concludes.

2 A model for the long-term volatility component

2.1 The time-varying GARCH framework

In this paper the tool for modelling an asset return series over a long period is a GARCH-type model. Finally, Section 6 concludes.

In order to account for changes in the long-run volatility we shall consider a more flexible specification in which the unconditional variance evolves smoothly over time. To motivate the introduction of our model we shall begin by focusing on the long-run properties of the GJR-GARCH(p, q) model of Glosten, Jagannathan, and Runkle (1993). Let \( h_{t-1} \) be the information set containing the historical information of the series of interest available at time \( t-1 \) and write the asset returns \( \{y_t\} \) as

\[
y_t = \mathbb{E}(y_t|\mathcal{F}_{t-1}) + \varepsilon_t
\]

where \( \{\varepsilon_t\} \) is a sequence of independent standard normal variables. Under this assumption the conditional distribution of the innovation sequence \( \{\varepsilon_t\} \) is \( \varepsilon_t|\mathcal{F}_{t-1} \sim N(0, h_t) \). For simplicity, the conditional mean of the asset returns is set equal to zero, i.e. \( \mathbb{E}(y_t|\mathcal{F}_{t-1}) = 0 \). The component \( h_t \) describes the dynamics of the conditional variance of the asset returns. To allow positive and negative shocks to have an asymmetric effect on the stock market volatility we choose the GJR-GARCH(p, q) model for \( h_t \). It has the form

\[
h_t = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{q} \kappa_i \varepsilon_{t-i}^2 I_{t-i}(\varepsilon_{t-i} < 0) + \sum_{j=1}^{p} \beta_j h_{t-j}. \tag{3}
\]

where the set of conditions for positivity and stationarity are imposed and \( I_{t-i}(\varepsilon_{t-i} < 0) \) is an indicator function that equals 1 when \( \varepsilon_{t-i} < 0, i = 1, \ldots, q \), and 0 otherwise. Re-writing the dynamic structure of (3) in terms of the unconditional variance \( \sigma^2 \) one obtains

\[
h_t = \sigma^2 + \sum_{i=1}^{q} \alpha_i (\varepsilon_{t-i}^2 - \sigma^2) + \sum_{i=1}^{q} \kappa_i (\varepsilon_{t-i}^2 I_{t-i}(\varepsilon_{t-i} < 0) - \sigma^2) + \sum_{j=1}^{p} \beta_j (h_{t-j} - \sigma^2). \tag{4}
\]

where \( \sigma^2 \equiv \mathbb{E}(\varepsilon_t^2) = \omega/(1 - \sum_{i=1}^{q} \alpha_i - \sum_{i=1}^{q} \kappa_i/2 - \sum_{j=1}^{p} \beta_j) \). When the persistence rate \( \sum_{i=1}^{q} \alpha_i + \sum_{i=1}^{q} \kappa_i/2 + \sum_{j=1}^{p} \beta_j < 1 \) then the conditional variance mean reverts to \( \sigma^2 \) at the geometric rate \( \sum_{i=1}^{q} \alpha_i + \sum_{i=1}^{q} \kappa_i/2 + \sum_{j=1}^{p} \beta_j \).

The assumption that the volatility process reverts to a constant level is very restrictive especially when modelling asset returns over long periods. In order to account for changes in the long-run volatility we shall consider a more flexible specification in which the unconditional variance \( \sigma^2 \) can be time-varying. We incorporate smooth changes in the unconditional variance of returns so that the variance evolves slowly over time. The variance is thus modelled using a multiplicative decomposition of the variance as follows:

\[
\varepsilon_t = \zeta_t^{1/2} g_t^{1/2}, \quad \varepsilon_t|\mathcal{F}_{t-1} \sim N(0, h_t g_t). \tag{5}
\]
In equation (5) the short-run (or the stationary) component \( h_t \) is modelled as the GJR-GARCH process as in (3) with the exception that \( \varepsilon_t^2 = \varepsilon_t^2 / g_t^{1/2} \):

\[
h_t = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{q} \kappa_i \varepsilon_{t-i}^2 I_{t-i}(\varepsilon_{t-i}^* < 0) + \sum_{j=1}^{p} \beta_j h_{t-j}.
\]

(6)

The long-run (or the nonstationary) component \( g_t \) is a slowly time-varying trend that functions as a proxy for all factors that affect the unconditional variance. More specifically, we follow \( ? \) and let the time-varying unconditional variance component be a linear combination of logistic transition functions:

\[
g_t = \delta_0 + \sum_{l=1}^{r} \delta_l G_l(t/T; \gamma_l, c_l)
\]

(7)

where \( \delta_l, l = 0, \ldots, r, \) are parameters. Furthermore, \( G_l(t/T; \gamma_l, c_l), l = 1, \ldots, r, \) are generalized logistic transition functions:

\[
G_l(t/T; \gamma_l, c_l) = \left( 1 + \exp \left\{ -\gamma_l \prod_{j=1}^{k} (t/T - c_{lj}) \right\} \right)^{-1}
\]

(8)

satisfying the identification restrictions \( \gamma_l > 0, l = 0, \ldots, r, \) and \( c_1 \leq c_2 \leq \ldots \leq c_k \). The transition functions \( G_l(t/T; \gamma_l, c_l) \) allow the unconditional variance to change smoothly as a function of the calendar time \( t/T \). The parameters, \( c_{lj} \) and \( \gamma_l \), determine the location and the speed of the transition between different regimes. Equations (5)–(8) define the time-varying GARCH (TV-GARCH) model. The unconditional variance in this model is time-varying and equals \( E_t(\varepsilon_t^2) = E(\varepsilon_t^2 h_t g_t) = g_t E h_t \). This approach of introducing nonstationarity in the long run volatility component has been discussed in detail by \( ? \).

Some special cases of the TV-GARCH model are of interest. Under \( \delta_1 = \ldots = \delta_r = 0 \), the unconditional variance \( E_t(\varepsilon_t^2) \) becomes constant. When \( r = 1 \) and \( k = 1 \), \( g_t \) increases (decreases) monotonically over time from \( \delta_0 \) to \( \delta_0 + \delta_1 \) when \( \delta_1 > 0(\delta_1 < 0) \), with the location centred at \( t = c_1 T \). The slope parameter \( \gamma_1 \) in (8) controls the degree of smoothness of the transition: the larger \( \gamma_1 \), the faster the transition is between the extreme regimes. When \( \gamma_1 \rightarrow \infty \), \( g_t \) collapses into a step function. For small values of \( \gamma_1 \), the transition between regimes is approximately linear around \( c_1 \). When \( \delta_1 \neq 0 \), for values \( r > 1 \) and \( k > 1 \), (7)–(8) form a very flexible parameterization capable of describing nonmonotonic deterministic changes in the unconditional variance.

### 2.2 Model specification

Since the nonlinear model in (5)–(8) is our most general parameterization, a systematic modelling strategy is required when a TV-GARCH model is fitted to the data. The strategy for building TV-GARCH models is based on statistical inference and it consists of the specification, estimation and evaluation of the model. At the specification stage, one first specifies the structure of \( g_t \) and, once that has been done, models the dynamics of the short-run component \( h_t \). In practice, the parametric structure of the unconditional variance component has to be determined from the data, which involves two sets of decision problems. First, the number of transitions \( r \) in (7) has to be determined. Second, when \( r \geq 1 \), the integer \( k \) for each transition function has to be selected. This specification procedure is sequential and based on statistical inference. We shall apply the procedure of \( ? \) for selecting \( r \) and \( k \).

An important feature of the modelling strategy in this paper is that, since we are modelling very long return series, we shall divide the observation period into a number of subperiods.
To introduce notation, let $r$ be the total number of transitions in the whole period and $r_i, i = 1, \ldots, N$, be the number of transitions in the subperiod $i$, so $r = \sum_{i=1}^{N} r_i$. Define $h_{it}$ as the conditional variance and $g_{it} = 1 + \sum_{l=1}^{r_i} \delta_{il} G_{it}(t/T; \gamma_{il}, c_{il}), i = 1, \ldots, N$, for each subperiod.

The sequence of LM tests for specifying a TV-GARCH model is as follows:

1. Split the original time series into $N$ non-overlapping subsamples. To facilitate specification the splits should preferably be located in tranquil periods.

2. For each $i = 1, \ldots, N$, specify $g_{it}$ under the assumption that the conditional variance is constant, i.e. $h_{it} \equiv \omega_i > 0$. This is done as follows. First, test the hypothesis of constant unconditional variance $H_{01} : \gamma_{i1} = 0$ against $H_{11} : \gamma_{i1} > 0$ in

$$g_{it} = \omega_i^{-1} \{1 + \delta_{i1} G_{i1}(t/T; \gamma_{i1}, c_{i1})\} = \omega_i^{-1} + \delta_{i1}^* G_{i1}(t/T; \gamma_{i1}, c_{i1})$$

where $\delta_{i1}^* = \omega_i^{-1} \delta_{i1}$, at the significance level $\alpha(1)$. Under the null hypothesis, the function contains unidentified nuisance parameters $\delta_{i1}$ and $c_{i1}$. To circumvent this identification problem we follow Luukkonen, Saikkonen, and Teräsvirta (1988) and approximate $G_{i1}(t/T; \gamma_{i1}, c_{i1})$ by its third-order Taylor expansion around $\gamma_{i1} = 0$. After reparameterizing, we obtain

$$g_{it} = \omega_i^* + \sum_{j=1}^{3} \phi_{ij}(t/T)^j + R_3(t/T; \gamma_{i1}, c_{i1})$$

where $\phi_{ij} = \gamma_{i1}^* \tilde{\delta}_{ij}, i = 1, \ldots, N$, and $R_3(t/T; \gamma_{i1}, c_{i1})$ is the remainder. Furthermore, $R_3(t/T; \gamma_{i1}, c_{i1}) \equiv 0$ under $H_{01}$, so the remainder of the Taylor expansion does not affect the asymptotic distribution theory. The null hypothesis of constant unconditional variance becomes $H'_{01} : \phi_{i1} = \phi_{i2} = \phi_{i3} = 0$. Under $H'_{01}$, the standard LM statistic has an asymptotic $\chi^2$-distribution with three degrees of freedom. See ?) for details on how to compute the test statistic.

3. If $H'_{01}$ is rejected, for each subperiod select the order $k \leq 3$ in the exponent of $G_{i1}(t/T; \gamma_{i1}, c_{i1})$ using a short sequence of tests within (10); for details see ?). Next, estimate $g_{it}$ with a single transition function and test $H_{02} : g_{it} = \omega_i^{-1} + \delta_{i1}^* G_{i1}(t/T; \gamma_{i1}, c_{i1})$ against $H_{12} : g_{it} = \omega_i^{-1} + \delta_{i1}^2 G_{i1}(t/T; \gamma_{i1}, c_{i1})$ at the significance level $\alpha(2) = \tau \alpha(1)$, where $\tau \in (0, 1)$. The significance level is reduced at each stage by a factor $\tau$ in order to favour parsimony. In our application we set $\tau = 0.5$. Test the hypothesis of no second transition $H_{02} : \gamma_{i2} = 0$ in

$$g_{it} = \omega_i^{-1} + \delta_{i1}^* G_{i1}(t/T; \gamma_{i1}, c_{i1}) + \delta_{i2}^* G_{i2}(t/T; \gamma_{i2}, c_{i2})$$

Again, model (11) is not identified under the null hypothesis. To circumvent the problem we proceed as before and express the logistic function $G_{i2}(t/T; \gamma_{i2}, c_{i2})$ by a third-order Taylor approximation around $\gamma_{i2} = 0$. After rearranging terms we have

$$g_{it} = \omega_i^* + \delta_{i1}^* G_{i1}(t/T; \gamma_{i1}, c_{i1}) + \sum_{j=1}^{3} \varphi_{ij}(t/T)^j + R_3(t/T; \gamma_{i2}, c_{i2})$$

where $\varphi_{ij} = \gamma_{i2}^* \tilde{\delta}_{ij}, i = 1, \ldots, N$, and $R_3(t/T; \gamma_{i2}, c_{i2})$ is the remainder. The new null hypothesis based on this approximation is $H'_{02} : \varphi_{i1} = \varphi_{i2} = \varphi_{i3} = 0$. Again, this hypothesis can be tested using a LM test. If the null hypothesis is rejected, specify $k$ for the second transition and estimate $g_{it}$ with two transition functions.

4. More generally, when $g_{it}$ has been estimated with $r_i - 1$ transition functions one tests for another transition in $g_{it}$ using the significance level $\alpha(\epsilon) = \tau \alpha(\epsilon-1)$. Testing continues until the first non-rejection of the null hypothesis.
In summary, we begin the model specification problem by first modelling the unconditional variance assuming that the conditional variances remain constant. After specifying and estimating $g_t$, the hypothesis of no conditional heteroskedasticity is tested in $\{\varepsilon_t^2\}$. If the null hypothesis of no ARCH is rejected, the conditional variance component $h_t$ is modelled as in (6) with $p = q = 1$. At the evaluation stage the adequacy of the estimated model is tested by means of LM-type misspecification tests (see Amado and Ter"asvirta (2011) for further details).

3 Estimation of parameters

After the number of transitions and their type in (7) have been determined, the parameters of the TV-GARCH model are estimated by quasi-maximum likelihood (QML). For this purpose, let $\theta = (\theta'_1, \theta'_2)'$ be the parameter vector of the model. Let $h_t \equiv h_t(\theta_1, \theta_2)$ and $g_t \equiv g_t(\theta_2)$ where $\theta_1 = (\omega, \alpha_1, \ldots, \alpha_q, \kappa_1, \ldots, \kappa_q, \beta_1, \ldots, \beta_p)'$ and $\theta_2 = (\delta', \gamma_1, \ldots, \gamma_r, c_1', \ldots, c_r')'$ with $\delta = (\delta_0, \delta_1, \ldots, \delta_r)'$. The model defined in (5)–(8) can be now rewritten as follows:

$$\varepsilon_t = \zeta_t \{h_t(\theta_1, \theta_2) g_t(\theta_2)\}^{1/2}.$$  \hspace{1cm} (13)

Assuming that $\{\zeta_t\}$ is a sequence of independent standard normal variables, the log-likelihood function for observation $t$ equals

$$\ell_t(\theta) = -(1/2) \ln 2\pi - (1/2) \{\ln h_t(\theta_1, \theta_2) + \ln g_t(\theta_2)\} - (1/2) \frac{\varepsilon_t^2}{h_t(\theta_1, \theta_2)g_t(\theta_2)}$$ \hspace{1cm} (14)

The unconditional and the conditional variance components are estimated separately using maximization by parts. The iterative algorithm proceeds as follows:

**Step 1:** Maximize

$$L_T^U(\theta_2) = \sum_{t=1}^T \ell_t^U(\theta_2) = -(1/2) \sum_{t=1}^T \{\ln g_t(\theta_2) + \varepsilon_t^2 / g_t(\theta_2)\}$$

with respect to $\theta_2$, assuming $\varepsilon_t = \varepsilon_t$, that is, setting $h_t(\theta_1, \theta_2) \equiv 1$. Let the estimator of $\theta_2$ be $\hat{\theta}_2^{(1)}$. Making use of $\hat{\theta}_2^{(1)}$, maximize

$$L_T^V(\theta_1, \hat{\theta}_2^{(1)}) = \sum_{t=1}^T \ell_t^V(\theta_1, \hat{\theta}_2^{(1)}) = -(1/2) \sum_{t=1}^T \{\ln h_t(\theta_1, \hat{\theta}_2^{(1)}) + \varepsilon_t^2 / h_t(\theta_1, \hat{\theta}_2^{(1)})\}$$

with respect to $\theta_1$, where $\varepsilon_t^* = \varepsilon_t / \{g_t(\hat{\theta}_2^{(1)})\}^{1/2}$. Denote the estimator as $\hat{\theta}_2^{(1)}$.

**Step 2:** Maximize

$$L_T^U(\theta_2) = \sum_{t=1}^T \ell_t^U(\theta_2) = -(1/2) \sum_{t=1}^T \{\ln g_t(\theta_2) + \varepsilon_t^2 / g_t(\theta_2)\}$$

with respect to $\theta_2$, where $\varepsilon_t = \varepsilon_t / \{h_t(\hat{\theta}_1^{(1)}, \hat{\theta}_2^{(1)})\}^{1/2}$. Call this estimator $\hat{\theta}_2^{(2)}$ and maximize

$$L_T^V(\theta_1, \hat{\theta}_2^{(2)}) = \sum_{t=1}^T \ell_t^V(\theta_1, \hat{\theta}_2^{(2)}) = -(1/2) \sum_{t=1}^T \{\ln h_t(\theta_1, \hat{\theta}_2^{(2)}) + \varepsilon_t^2 / h_t(\theta_1, \hat{\theta}_2^{(2)})\}$$

with respect to $\theta_1$, where $\varepsilon_t^* = \varepsilon_t / \{g_t(\hat{\theta}_2^{(2)})\}^{1/2}$. This yields $\hat{\theta}_1^{(2)}$.
Iterate until convergence.

In the \(n\)th iteration, maximization is carried out in the usual way by solving the score equations:

\[
\frac{\partial}{\partial \theta_2} L^U_T(\theta_2) = (1/2) \sum_{t=1}^{T} \left( \frac{\hat{\varepsilon}_t^2}{g_t(\theta_2)} - 1 \right) \frac{1}{g_t(\theta_2)} \frac{\partial g_t(\theta_2)}{\partial \theta_2} = 0
\]

for \(\theta_2\) with \(\hat{\varepsilon}_t = \varepsilon_t / \{h_t(\hat{\theta}_1^{(n-1)}, \hat{\theta}_2^{(n-1)})\}^{1/2}\), and

\[
\frac{\partial}{\partial \theta_1} L^V_T(\theta_1) = (1/2) \sum_{t=1}^{T} \left( \frac{\varepsilon_t^2}{h_t(\theta_1, \hat{\theta}_2^{(n)})} - 1 \right) \frac{1}{h_t(\theta_1, \hat{\theta}_2^{(n)})} \frac{\partial h_t(\theta_1, \hat{\theta}_2^{(n)})}{\partial \theta_1} = 0
\]

for \(\theta_1\), where \(\varepsilon_t^* = \varepsilon_t / \{g_t(\hat{\theta}_2^{(n)})\}^{1/2}\). Letting \(G_{lt} \equiv G_l(t/T; \gamma_l, c_l), l = 1, \ldots, r\), we have

\[
\frac{\partial g_t(\theta_2)}{\partial \theta_2} = (1, G_{lt}, G_{lt}^{(c)}, G_{lt}^{(g)}; \ldots, G_{lt}, G_{lt}^{(c)})'
\]

where, for \(k = 1\) in (8),

\[
G_{lt}^{(c)} = \frac{\partial G_{lt}}{\partial \gamma_l} = \delta_l G_{lt}(1 - G_{lt})(t/T - c_l)
\]

\[
G_{lt}^{(g)} = \frac{\partial G_{lt}}{\partial c_l} = -\gamma_l \delta_l G_{lt}(1 - G_{lt})
\]

and for \(k > 1\)

\[
G_{lt}^{(c)} = \frac{\partial G_{lit}}{\partial c_l} = -\gamma_l \delta_l G_{lt}(1 - G_{lt}) \prod_{j=1, j \neq l}^{k} (t/T - c_j)
\]

where \(c_{lj}\) denotes the \(j\)th element in the parameter vector \(c_l, l = 1, \ldots, r\), and

\[
\frac{\partial h_t(\theta_1, \hat{\theta}_2^{(n)})}{\partial \theta_1} = (1, \varepsilon_{t-1}^*, \ldots, \varepsilon_{t-q}^*, \varepsilon_{t-1}^* I_{t-1}(\varepsilon_{t-1}^* \varepsilon_{t-1}^* < 0), \ldots, \varepsilon_{t-q}^* I_{t-q}(\varepsilon_{t-q}^* < 0),
\]

\[
\begin{align*}
&h_{t-1}(\theta_1, \hat{\theta}_2^{(n)}), \ldots, h_{t-p}(\theta_1, \hat{\theta}_2^{(n)})' + \sum_{j=1}^{p} \beta_j \frac{\partial h_{t-j}(\theta_1, \hat{\theta}_2^{(n)})}{\partial \theta_1} \end{align*}
\]

This algorithm is computationally attractive for situations in which direct maximization of the log-likelihood function is difficult. Under certain regularity conditions, the resulting estimator coincides with the ML estimator and becomes fully efficient upon convergence; see Song, Fan, and Kalbfleisch (2005) for details. Throughout this paper, we assume that certain regularity conditions are satisfied to ensure consistency and asymptotic normality of the QML estimator. The asymptotic properties of the estimators of the TV-GARCH model are not yet known. Extending the results to the nonstationary TV-GARCH model is not straightforward and is beyond the scope of this paper.

In this work, the long time series requires some modifications to the estimation algorithm. Because the whole series is divided into non-overlapping subperiods, the different data segments can have different “baseline” volatility levels. For this reason, the algorithm iterates from an initial value which is estimated by “chain rule” to accommodate differences in the volatility.
levels. This proceeds as follows. First, for the first subperiod, estimate the parameters of \( g_{1t} = \delta_0 + \sum_{l=1}^{r_1} \delta_l G_{1l}(t/T; \gamma_{1l}, c_{1l}) \) where \( r_1 \) is the number of transitions for this period. The estimate \( \hat{g}_{1t} \) serves as the “intercept” in the nonstationary component of the next subperiod. Conditioning on this value, carry out the estimation of the parameters for the next subperiod. More generally, for the \( i \)th subperiod, estimate \( g_{it} = \hat{\delta}_0^{(i-1)} + \sum_{l=1}^{r_i} \delta_l G_{il}(t/T; \gamma_{il}, c_{il}) \) by conditioning on \( \hat{\delta}_0^{(i-1)} \), where \( \hat{\delta}_0^{(i-1)} = \hat{\delta}_0 + \sum_{l=1}^{r_{i-1}} \delta_l G(t/T; \hat{\gamma}_l, \hat{c}_l) \) and \( r_{i-1} \) is the number of transitions in the \((i - 1)\)th subperiod. The estimates \( \hat{\gamma}_l \) and \( \hat{c}_l \) are then used as fixed values in the next iterations. This means that the estimation algorithm is carried out without iterating \( \hat{\gamma}_l \) and \( \hat{c}_l \), and therefore the parameters \( \delta_l, l = 0, \ldots, r \), are estimated conditionally on those estimates.

Another aspect that deserves attention in the estimation of the model is the selection of starting-values of the time-varying parameters. Since the log-likelihood may contain several local maxima, it is advisable to initiate the estimation from different sets of starting-values before settling for the final parameter estimates. In addition, to improve the accuracy of the estimates of the standard errors, we follow Fiorentini, Calzolari, and Panattoni (1996) and use analytic first derivatives both in the estimation of the TV-GARCH models and in the computation of the test statistics. All computations in this paper have been carried out using Ox programming language, version 3.40 (see Doornik (2002)).

4 Application to the Dow Jones Industrial Average index

4.1 Data description

In this section we illustrate the use of the modelling building procedure of the TV-GARCH model to the daily returns of the Dow Jones Industrial Average (DJIA) index. The entire sample covers the period between January 2, 1920 and December 31, 2003, yielding 21121 observations. The daily returns are defined as the log differences of the closing prices of the index between two consecutive days. The closing prices of the DJIA index have been obtained from the Wharton Research Data Services (WRDS) provided by the Wharton School of the University of Pennsylvania. Descriptive statistics of the return series can be found in Table 1. The coefficients of skewness and kurtosis seem to indicate that the stock returns \( \varepsilon_t \) have a left skewed and a significantly fat-tailed distribution. To check this conclusion, we also provide the robust measures of skewness and kurtosis as recommended by Kim and White (2004) in order to account for outliers. The robust measure for skewness is practically zero whereas the robust kurtosis measure suggests that there is indeed some excess kurtosis in the series. Figure 1 graphs the daily returns for the DJIA index for the observation period. The period covers the Great Depression of 1929 and the early 1930’s, the Second World War, the 1973 oil crisis, the stock market crash of October 1987 and the recent dot-com bubble. Because of the long observation period it is unlikely that the series is stationary.

We divide the 83 years long series into six non-overlapping subperiods each comprising at least of 2500 observations. In most cases we report the findings for each of the six periods and the full sample. Summary statistics of the subperiods can be found in Table 9.

| Table 1: Summary statistics of the daily DJIA return series: full sample |
|------------------|---|---|---|---|---|---|---|
| \( \varepsilon_t \) | -25.63 | 14.27 | 0.022 | 1.136 | -0.659 | 25.28 | -0.006 | 0.227 |
| \( \varepsilon_t/\hat{g}_{1/2} \) | -17.71 | 6.680 | 0.018 | 0.660 | -1.185 | 29.73 | -0.006 | 0.145 |

Notes: The table contains summary statistics for the DJIA return series. The sample period starts in January 2, 1920 and ends in December 31, 2003 (21121 observations).
Figure 1: The DJIA daily returns from January 2, 1920 until December 31, 2003. The vertical lines represent the split dates.

4.2 Estimation results

The focus of the empirical analysis lies in the specification of the unconditional variance using the modelling strategy described in Section 2.2. We begin by determining the number of transitions for each subperiod separately. This is done using the sequence of specification tests. The initial significance level of the sequence of tests is $\alpha^{(1)} = 0.01$. At each stage of the sequence we halve the significance level of the test, i.e. $\tau = 0.5$. The tests results are presented in the second column of Table 2.

We first test the hypothesis of constant unconditional variance against a smoothly time-varying unconditional variance with one transition function. The null hypothesis is rejected for all subperiods with the exception of the subperiod 5 covering the October 1987 crash. The stock market volatility returned to normal levels very quickly after the crash, which suggests that the unconditional variance remained stable during that period. These findings are consistent with the hypothesis of Engle and Lee (1999) that the 1987 crash is more transient than other big shocks. The null hypothesis of constant unconditional variance is, however, rejected very strongly for the subperiods 1, 2, 4 and 6. The first period contains the Great Depression, the second includes the Second World War, the fourth one the OPEC oil crisis and the most recent one the IT bubble. The results indicate that the strongest deterministic changes in the unconditional variance are associated with the largest economic recessions in the period under study.

The sequence of nested tests based on (10) to select $k$ in (8) are given in the last three columns of Table 2 (see Amado and Teräsvirta (2011) for details). The strongest rejection is when $k = 1$ for all four periods. Note that, for the subperiod 1, the tests $H_{01}$ and $H_{02}$ cannot
discriminate between \(k = 1\) and \(k = 2\) as the \(p\)-values are very close to each other. We choose \(k = 2\) to minimize the number of transitions to be specified. Misspecification tests to check the validity of this choice can be carried out at the model evaluation stage.

We proceed to first estimate a TV-GARCH model with a single transition and test against a double transition model at \(\alpha^{(2)} = 0.005\). We reject the hypothesis in three out of the five cases and select \(k = 1\). Fitting the model with two transition functions for the three subsamples and testing against another transition leads to a rejection only for the fourth period. The \(p\)-value, however, is 0.0021 which is very close to \(\alpha^{(3)} = 0.0025\). Thus, we tentatively accept the model with two transitions as the final parameterization for the first, second and fourth subperiods.

The above results imply that eight transition functions in total are needed to describe the unconditional variance for the whole series. Estimation results for the TV-GARCH model are reported in Table 3, Panel (a). The estimation results for each of the subperiods can be found in Table 10 in Appendix B.

The estimation is carried out with the sequential quadratic programming optimisation algorithm using analytical derivatives. The numbers in parenthesis below the parameter estimates are the asymptotic standard error estimates and calculated using numerical second derivatives. The standard errors of \(\gamma_j\) and \(c_i, i = 1, \ldots, 8\), are not available because the parameters \(\delta_j, j = 0, \ldots, 8\), are estimated conditionally on those parameters. In some subperiods we observe that the transition between the extreme regimes of volatility is quite rapid. For these cases, the maximum value of \(\gamma_j\) is constrained to 100 to avoid convergence problems. This approximation is adequate because the shape of the transition function does not change much beyond values of \(\gamma_j\) larger or equal than 100.

For an idea how the unconditional variance changes over time, the estimated component \(g_t^{1/2}\) is plotted in Figure 2 (upper panel). The estimated \(g_t\) functions for each subperiod are also shown in Figure 8 (Appendix A). It is seen that the largest deterministic changes in the unconditional variance occur during the periods of recession in the economy. In particular, the strongest movement in the long-run volatility is observed during the Great Depression. This is in agreement with Mikosch and St˘aric˘a (2004) who find that most of the recessions coincide with an increase in the unconditional variance of the series. In their analysis of the S&P 500 returns, they identify the 1973 oil crisis as the major change detected in the unconditional variance, but then they study a time series only covering the period from January 2, 1953, until December 31, 1990.

For comparison, we also report the results of fitting the GJR-GARCH(1,1) model into the complete series. They can be found in Panel (b) of Table 3. The results for each subperiod appear in Table 11 (Appendix B). We find that the subperiods characterized by the largest changes in the unconditional variance have a stronger integrated GJR-GARCH effect. The stationary condition for the full sample model is \(\hat{\alpha}_1 + \hat{\beta}_1 + \hat{\kappa}_1/2 < 1\). This model is practically an integrated GJR-GARCH model as the persistence indicator \(\hat{\alpha}_1 + \hat{\beta}_1 + \hat{\kappa}_1/2 = 0.9903\). The autocorrelation functions of \(|\epsilon_t|\) plotted in Figure 3 (upper panel) lead to the same conclusion. The graph clearly displays the long-memory property: relatively rapid decay at short lags followed by positive autocorrelations around a stable level at long lags. On the contrary, the autocorrelations of \(|\epsilon_t|/\hat{g}_t^{1/2}\), plotted in the lower panel of Figure 3, decay very quickly with the lag length and only the first 70 autocorrelation estimates or so are significantly different from zero judging from the 95% confidence bounds drawn under the assumption that the errors are normal and independent. The decay rate looks more or less exponential, and the persistence indicator now equals 0.97. The results show that modelling the changes in the unconditional variance strongly reduces the amount of evidence for long-memory. This can also be seen from the Geweke and Porter-Hudak (1983) (GPH) estimates of the long-memory parameter in Table 4. Of course, the GPH parameter estimates are different for different bandwidths but, overall, the table indicates that the daily DJIA return series is either nonstationary (for the bandwidth choices \(m = T^{0.4}\).
Figure 2: First panel, shows the estimated function $g_t^{1/2}$ (black curve) and the conditional standard deviation from the GJR-GARCH(1,1) model (grey curve). Second panel, shows the estimated conditional standard deviations from the GJR-GARCH(1,1) model (grey curve) and from the TV-GJR-GARCH(1,1) (black curve) models.
Table 2: *p*-values of sequences of Lagrange multiplier tests for the six subperiods

<table>
<thead>
<tr>
<th>Subperiods</th>
<th>$H_0$</th>
<th>$H_{03}$</th>
<th>$H_{02}$</th>
<th>$H_{01}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Subsample 1 (02/01/1920 – 31/12/1931)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Single transition</td>
<td>$5 \times 10^{-44}$</td>
<td>$7 \times 10^{-4}$</td>
<td>$3 \times 10^{-22}$</td>
<td>$3 \times 10^{-24}$</td>
</tr>
<tr>
<td>Double transition</td>
<td>$2 \times 10^{-8}$</td>
<td>0.4773</td>
<td>0.0111</td>
<td>$2 \times 10^{-8}$</td>
</tr>
<tr>
<td>Triple transition</td>
<td>0.2632</td>
<td>0.0574</td>
<td>0.7178</td>
<td>0.6213</td>
</tr>
<tr>
<td><strong>Subsample 2 (04/01/1932 – 31/12/1943)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Single transition</td>
<td>$5 \times 10^{-77}$</td>
<td>$3 \times 10^{-19}$</td>
<td>$3 \times 10^{-20}$</td>
<td>$2 \times 10^{-46}$</td>
</tr>
<tr>
<td>Double transition</td>
<td>$2 \times 10^{-6}$</td>
<td>0.0017</td>
<td>0.0242</td>
<td>$1 \times 10^{-4}$</td>
</tr>
<tr>
<td>Triple transition</td>
<td>0.2818</td>
<td>0.1547</td>
<td>0.3484</td>
<td>0.3388</td>
</tr>
<tr>
<td><strong>Subsample 3 (04/01/1944 – 29/12/1961)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Single transition</td>
<td>0.0079</td>
<td>0.0712</td>
<td>0.9654</td>
<td>0.0034</td>
</tr>
<tr>
<td>Double transition</td>
<td>0.0792</td>
<td>0.2789</td>
<td>0.2364</td>
<td>0.0402</td>
</tr>
<tr>
<td>Triple transition</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td><strong>Subsample 4 (01/01/1962 – 16/11/1982)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Single transition</td>
<td>$1 \times 10^{-17}$</td>
<td>$2 \times 10^{-4}$</td>
<td>0.3328</td>
<td>$3 \times 10^{-16}$</td>
</tr>
<tr>
<td>Double transition</td>
<td>$4 \times 10^{-5}$</td>
<td>$9 \times 10^{-4}$</td>
<td>0.8199</td>
<td>$6 \times 10^{-4}$</td>
</tr>
<tr>
<td>Triple transition</td>
<td>0.0021</td>
<td>0.1889</td>
<td>0.0006</td>
<td>0.2988</td>
</tr>
<tr>
<td><strong>Subsample 5 (17/11/1982 – 31/12/1993)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Single transition</td>
<td>0.1018</td>
<td>0.8983</td>
<td>0.0173</td>
<td>0.4688</td>
</tr>
<tr>
<td>Double transition</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td><strong>Subsample 6 (03/01/1994 – 31/12/2003)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Single transition</td>
<td>$3 \times 10^{-18}$</td>
<td>0.0012</td>
<td>0.0017</td>
<td>$8 \times 10^{-16}$</td>
</tr>
<tr>
<td>Double transition</td>
<td>0.0315</td>
<td>0.0361</td>
<td>0.0694</td>
<td>0.2817</td>
</tr>
<tr>
<td>Triple transition</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>--</td>
</tr>
</tbody>
</table>

**Notes:** The entries are the *p*-values of the LM-type tests of constant unconditional variance against a time-varying GARCH model for each subperiod of the DJIA stock index returns. The test sequence starts at the significance level $\alpha = 0.01$ and setting $\tau = 0.5$. The order $k$ in (8) is chosen from the sequence of nested tests based on (10). If $H_{0i}$ is rejected most strongly, measured by the *p*-value, of the three hypotheses, one selects $k = i$. See ?) for further details.
Table 3: Estimation results for the DJIA returns: full sample

Panel (a): parameter estimates of the TV-GJR-GARCH(1,1) model

\[
\begin{align*}
    h_t &= 0.0285 + 0.0236 \varepsilon_{t-1}^2 + 0.9011 h_{t-1} + 0.0913 I_{t-1}(\varepsilon_{t-1}^* < 0) \varepsilon_{t-1}^2 \\
    \log\text{-Lik} &= -19112.1 \\
    \hat{\alpha}_1 + \hat{\beta}_1 + \hat{\kappa}_1/2 &= 0.9704
\end{align*}
\]

\[
\begin{align*}
    g_t &= 0.82506 + 1.1680(1 + \exp\{-52.81(t/T - 0.0119)(t/T - 0.1047)\})^{-1} \\
    &\quad + 6.3472(1 + \exp\{-100(t/T - 0.1154)\})^{-1} \\
    &\quad - 6.4953(1 + \exp\{-96.65(t/T - 0.1643)\})^{-1} \\
    &\quad - 1.2475(1 + \exp\{-70.03(t/T - 0.2454)\})^{-1} \\
    &\quad - 0.1522(1 + \exp\{-100(t/T - 0.3697)\})^{-1} \\
    &\quad + 1.7541(1 + \exp\{-67.17(t/T - 0.6355)\})^{-1} \\
    &\quad - 1.2656(1 + \exp\{-100(t/T - 0.6544)\})^{-1} \\
    &\quad + 0.6952(1 + \exp\{-100(t/T - 0.9216)\})^{-1}
\end{align*}
\]

Panel (b): parameter estimates of the GJR-GARCH(1,1) model

\[
\begin{align*}
    h_t &= 0.0116 + 0.0304 \varepsilon_{t-1}^2 + 0.9208 h_{t-1} + 0.0784 I_{t-1}(\varepsilon_{t-1}^* < 0) \varepsilon_{t-1}^2 \\
    \log\text{-Lik} &= -27468.5 \\
    \hat{\alpha}_1 + \hat{\beta}_1 + \hat{\kappa}_1/2 &= 0.9903
\end{align*}
\]
Figure 3: First panel, shows the sample autocorrelation functions of the absolute values of the DJIA daily returns. Second panel, shows the sample autocorrelation functions of the standardized variable $|\epsilon_t|/\hat{g}_t^{1/2}$. The horizontal lines are the corresponding 95% confidence interval under the iid normality assumption.
and \( m = T^{0.5} \) or is very close to the nonstationary region (for \( m = T^{0.6} \)). However, when the movements in the unconditional variance component are taken into account the GPH estimates have the remarkable low values of 0.1198, 0.2340 and 0.3050 for these three bandwidths.

### Table 4: GPH estimates of the long-memory parameter: full sample

<table>
<thead>
<tr>
<th>( d_{GPH}(m = T^{0.4}) )</th>
<th>( d_{GPH}(m = T^{0.5}) )</th>
<th>( d_{GPH}(m = T^{0.6}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varepsilon_t )</td>
<td>0.7364</td>
<td>0.5470</td>
</tr>
<tr>
<td></td>
<td>(0.0614)</td>
<td>(0.0511)</td>
</tr>
<tr>
<td>( \varepsilon_t/\hat{g}_t^{1/2} )</td>
<td>0.1198</td>
<td>0.2340</td>
</tr>
<tr>
<td></td>
<td>(0.1035)</td>
<td>(0.0576)</td>
</tr>
</tbody>
</table>

Notes: The numbers in parentheses are the standard errors. The bandwidth \( m \) equals \( T^\alpha, \alpha \in \{0.4, 0.5, 0.6\} \) where \( T \) is the number of observations.

A similar conclusion can be reached by looking at the estimated conditional at the estimated conditional standard deviations from the GJR-GARCH(1,1) model of \( \varepsilon_t \) and \( \varepsilon_t/\hat{g}_t^{1/2} \). The lower panel of Figure 2 displays both series. The (almost) stationary behaviour of the conditional standard deviation of \( \varepsilon_t/\hat{g}_t^{1/2} \) (black curve) contrasts with the nonstationary behaviour of the conditional standard deviation of \( \varepsilon_t \) (grey line). It shows that the conditional variance of \( \varepsilon_t/\hat{g}_t^{1/2} \) is considerably smaller than that of \( \varepsilon_t \) from the GJR-GARCH(1,1) model. For illustration, we also show in Figure 9 (Appendix A) the estimated conditional standard deviations generated from both models separately for each subperiod.

The adequacy of the estimated TV-GJR-GARCH(1,1) model is checked using the diagnostic tests proposed by Amado and Teräsvirta (2011). We perform tests against remaining ARCH in the standardized residuals, additional transitions in \( g_t \), TV-GJR-GARCH(1,2) and TV-GJR-GARCH(2,1) models, and ST-GJR-GARCH(1,1) model of order 1. The \( p \)-values of the tests are given in Tables 5-7. For comparison we also show the test results for the estimated GJR-GARCH(1,1) model. The results indicate no evidence of remaining ARCH in the standardized residuals, nor can argue in favour of additional transitions in \( g_t \); see Tables 5 and 6. However, the tests against TV-GJR-GARCH(1,2) and TV-GJR-GARCH(2,1) reject the null hypothesis at the 5% significance level; see Table 7. Moreover, the TV-GJR-GARCH(1,1) model is strongly rejected against ST-GJR-GARCH(1,1) model. The results suggests that the TV-GJR-GARCH(1,1) model is an inadequate parameterization, and a higher lag in the GJR-GARCH component or a nonlinear GARCH model should be employed. Modelling the short-run dynamics of volatility over a long time series does need more work. But then, the focus of this paper is on the modelling of changes in the long-run volatility component and refinements in the modelling of \( h_t \) are left for further work.

### 5 Monte Carlo experiment

In this section, we further investigate the effects of ignoring deterministic changes in the unconditional variance on the estimation of two GARCH-type models. This is done by conducting a small Monte-Carlo experiment. The purpose of the experiment is to illustrate how such shifts may bias the GARCH parameters and the persistence indicator. We generate data from two models. The first model is a TV-GARCH(1,1) model, whereas the second one is a TV-GJR-GARCH(1,1) model. The data-generating process is defined as follows:

\[
\varepsilon_t = \zeta_t h_t^{1/2} g_t^{1/2}, \quad \varepsilon_t | F_{t-1} \sim N(0, \hat{g}_t g_t) \tag{15}
\]

\[
h_t = \omega + \alpha_1 \varepsilon_{t-1}^2 + \kappa_1 \varepsilon_{t-1}^2 I_{t-1}(\varepsilon_{t-1} < 0) + \beta_1 h_{t-1} \tag{16}
\]

\[
g_t = 1 + \delta_1 (1 + \exp(-\gamma_1(t/T - c_1)))^{-1} \tag{17}
\]
with $\omega = 0.05, \alpha_1 = \{0.1, 0.05\}, \kappa_1 = \{0, 0.05\}, \beta_1 = 0.8, \delta_1 = \{0.05, 0.10\}$, and $c_1 = 0.5$ in each experiment. The simulations differ according to the pair $\{\alpha_1, \kappa_1\}$ and to the values of the slope parameter $\gamma_1$ which varies in the interval $\gamma_1 = \{10, 50\}$. The first 1000 observations of each generated series have been discarded to avoid initialization effects. For each experiment, the number of replications equals 2000 with a sample size of 5000 observations.

Figure 4 contains the estimated density of the estimated GARCH parameters and the persistence measured by the sum $\hat{\alpha}_1 + \hat{\beta}_1$ when the true model is (15)-(17) with $\{\alpha_1, \kappa_1\} = \{0.1, 0\}$ and $\gamma_1 = 10$. The figure shows that the probability mass of the empirical distribution of the parameter $\omega$ is shifted to the left and is very close to zero and that the empirical distribution of the parameter $\beta_1$ is shifted to the right when $\delta_1 = 0.05$. These results are even more striking for a large change in the unconditional variance, i.e. for $\delta_1 = 0.1$. The empirical distribution of the persistence of volatility shocks measured by $\hat{\alpha}_1 + \hat{\beta}_1$ is very close to 0.95 when small changes occur in the unconditional variance. Of course, the probability of $\hat{\alpha}_1 + \hat{\beta}_1$ being very close to one increases with the size of the deterministic change. For changes in the unconditional variance well approximated by a step function, the bias in measuring the persistence of volatility shocks is particularly severe. The plots of the estimated densities for $\gamma_1 = 50$ can be found in Figure 5. These findings are in agreement with the well documented results in the GARCH literature that shifts in the unconditional variance lead to an upward bias in the persistence of volatility shocks (see e.g. Lamoureux and Lastrapes (1990)).

In the second design, data generated from a TV-GJR-GARCH(1,1) model for $\{\alpha_1, \kappa_1\} = \{0.05, 0.05\}$ is fitted to a GJR-GARCH(1,1) model. The results are presented in Figures 6-7. The findings in the asymmetric model are very similar to the symmetric case. The empirical distributions of the parameters $\omega$ and $\kappa_1$ are shifted to the left and that the empirical distributions of the parameters $\alpha_1$ and $\beta_1$ are shifted to the right due to changes in the unconditional variance. Again, the probability of finding the persistence indicator $\hat{\alpha}_1 + \hat{\kappa}_1/2 + \hat{\beta}_1$ very close to one is very large. This effect is more remarkable when the changes in the unconditional variance are abrupt. Moreover, the simulations agree with the results obtained in the application.

It may be of interest to investigate the behaviour of the LM-type tests involved in the modelling strategy when the data has been generated by a long-memory process. For this purpose, we study the empirical power properties of the test statistic using as data-generating process the FIGARCH model of Baillie, Bollerslev, and Mikkelsen (1996). In particular, the artificial series is generated according to a FIGARCH(1,d,1) model as follows:

$$
\varepsilon_t = \zeta h_t^{1/2}, \quad \varepsilon_t | \tilde{\varepsilon}_{t-1} \sim N(0, h_t)
$$

$$
[1 - \beta L] h_t = \omega + [1 - \beta L - \phi L(1 - L)^d] \varepsilon_t^2
$$

Table 5: Diagnostic tests: $p$-values of the test of no ARCH in GARCH

<table>
<thead>
<tr>
<th>Model</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>GJR-GARCH(1,1)</td>
<td>0.197</td>
<td>0.266</td>
<td>0.448</td>
<td>0.612</td>
<td>0.718</td>
</tr>
<tr>
<td>TV-GJR-GARCH(1,1)</td>
<td>0.366</td>
<td>0.360</td>
<td>0.561</td>
<td>0.717</td>
<td>0.835</td>
</tr>
</tbody>
</table>

Table 6: Diagnostic tests: $p$-values of tests of no additional transition in $g_t$

<table>
<thead>
<tr>
<th>Model</th>
<th>LM1</th>
<th>LM2</th>
<th>LM3</th>
</tr>
</thead>
<tbody>
<tr>
<td>GJR-GARCH(1,1)</td>
<td>0.905</td>
<td>$4 \times 10^{-5}$</td>
<td>$1 \times 10^{-4}$</td>
</tr>
<tr>
<td>TV-GJR-GARCH(1,1)</td>
<td>0.092</td>
<td>0.186</td>
<td>0.321</td>
</tr>
</tbody>
</table>
Table 7: Diagnostic tests: p-values of tests against models of higher orders and against a non-linear structure

<table>
<thead>
<tr>
<th>Alternative model</th>
<th>GJR(1,2)</th>
<th>GJR(2,1)</th>
<th>ST-GJR ($K = 1$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>GJR-GARCH(1,1)</td>
<td>0.0094</td>
<td>0.006</td>
<td>$1 \times 10^{-6}$</td>
</tr>
<tr>
<td>TV-GJR-GARCH(1,1)</td>
<td>0.0086</td>
<td>0.036</td>
<td>$3 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

Table 8: Actual rejection frequencies of the standard LM test of constant unconditional variance

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\alpha$</th>
<th>$T = 1000$</th>
<th>$T = 2500$</th>
<th>$T = 5000$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>LM$_1$</td>
<td>LM$_3$</td>
<td>LM$_1$</td>
</tr>
<tr>
<td>1%</td>
<td></td>
<td>18.20</td>
<td>27.80</td>
<td>19.55</td>
</tr>
<tr>
<td>5%</td>
<td></td>
<td>33.15</td>
<td>48.35</td>
<td>33.85</td>
</tr>
<tr>
<td></td>
<td></td>
<td>27.80</td>
<td>48.35</td>
<td>33.85</td>
</tr>
<tr>
<td>1%</td>
<td></td>
<td>29.85</td>
<td>49.20</td>
<td>29.60</td>
</tr>
<tr>
<td>5%</td>
<td></td>
<td>45.25</td>
<td>67.55</td>
<td>44.75</td>
</tr>
<tr>
<td></td>
<td></td>
<td>38.30</td>
<td>62.10</td>
<td>30.40</td>
</tr>
<tr>
<td>1%</td>
<td></td>
<td>37.65</td>
<td>66.30</td>
<td>35.50</td>
</tr>
<tr>
<td>5%</td>
<td></td>
<td>50.95</td>
<td>78.80</td>
<td>48.15</td>
</tr>
<tr>
<td></td>
<td></td>
<td>42.45</td>
<td>69.65</td>
<td>32.65</td>
</tr>
</tbody>
</table>

Notes: Monte Carlo results in percentage of the non-robust LM parameter constancy test based on 2000 replications. The artificial series is generated according to a FI-GARCH(1,d,1) model $y_t = \varepsilon_t, \varepsilon_t|\mathcal{F}_{t-1} \sim N(0, h_t)$ and $|1 - \beta L|h_t = \omega + |1 - \beta L - \phi L(1 - L)^d|\varepsilon_t^2$. Results are shown for the LM test at the 1%, 5% and 10% nominal significance levels. The columns ‘LM$_1$’ and ‘LM$_3$’ correspond to the test procedure based on the first-order and third-order Taylor expansions, respectively.

where $\omega = 0.05, \phi = 0.5$ and $\beta = 0.7$. We use sample sizes of 1000, 2500 and 5000 observations with 2000 replications. The actual rejection frequencies of the test at 1% and 5% critical values are reported in Table 8. Since we focus on the power results when the long-memory parameter $d$ is located in the nonstationary region we use $d = 0.5, 0.6, 0.7$ and 0.8. The power of the tests is moderate for the LM$_1$—type test for the sample size of 2500 observations. The LM$_3$—type test is, however, more powerful than the LM$_1$—test and it has better power properties for the larger sample size. Interestingly, the power is higher at low values of the long memory parameter $d$ it is at higher values of this parameter.

6 Conclusions

In this paper we develop a testing and modelling procedure for describing the long-term movements in stock market returns over very long time periods. This is done by multiplicatively decomposing the variance of a GARCH model into a conditional and an unconditional component, in which the unconditional variance is allowed to change smoothly over time. The proposed model is the Time-Varying GARCH model as in Amado and Ter"asvirta (2011). The model building strategy relies on statistical inference, making use of a sequence of Lagrange-multiplier type specification tests. Because of the length of the observation period, the time series is divided into non-overlapping subperiods with the aim of alleviating the model building procedure. One advantage of this device is that it provides a modelling framework particularly useful in applications involving very long time series.

An empirical example applied to the long daily DJIA return series shows how the technique works in practice. Our results show that the dependence structure of the series is best explained
Figure 4: Estimated densities of the estimated GARCH(1,1) parameters when the DGP is a TV-GARCH(1,1) model with a single transition. The observations are generated by the process $\varepsilon_t = \zeta_t h_t^{1/2} \gamma_t^{1/2}$ where $h_t = 0.05 + 0.1\varepsilon_{t-1}^2 + 0.8h_{t-1}$ and $g_t = 1 + \delta_1 (1 + \exp\{-10(t/T - 0.5)\})^{-1}$. The results are based on 2000 replications with 5000 observations.
Figure 5: Estimated densities of the estimated GARCH(1,1) parameters when the DGP is a TV-GARCH(1,1) model with a single transition. The observations are generated by the process $\varepsilon_t = \zeta_t h_t^{1/2} g_t^{1/2}$ where $h_t = 0.05 + 0.1 \varepsilon_{t-1}^2 + 0.8 h_{t-1}$ and $g_t = 1 + \delta_1 (1 + \exp(-50(t/T - 0.5)))^{-1}$. The results are based on 2000 replications with 5000 observations.
The process $\varepsilon_t = \zeta h_{t-1}^{1/2} g_t^{1/2}$ where $h_t = 0.05 + 0.05\varepsilon_{t-1}^2 + 0.1\varepsilon_{t-1}^2 I_{t-1}(\varepsilon_{t-1} < 0) + 0.8h_{t-1}$ and $g_t = 1 + \delta_t (1 + \exp\{-10(t/T - 0.5)\})^{-1}$. The results are based on 2000 replications with 5000 observations.

Figure 7: Estimated densities of the estimated GJR-GARCH(1,1) parameters when the DGP is a TV-GJR-GARCH(1,1) model with a single transition. The observations are generated by the process $\varepsilon_t = \zeta h_{t-1}^{1/2} g_t^{1/2}$ where $h_t = 0.05 + 0.05\varepsilon_{t-1}^2 + 0.1\varepsilon_{t-1}^2 I_{t-1}(\varepsilon_{t-1} < 0) + 0.8h_{t-1}$ and $g_t = 1 + \delta_t (1 + \exp\{-50(t/T - 0.5)\})^{-1}$. The results are based on 2000 replications with 5000 observations.
by deterministic changes in the unconditional variance, and consequently the hypothesis of constant unconditional variance turns out to be inappropriate. We also show empirically and with a small Monte Carlo experiment how unmodelled deterministic changes in the unconditional variance reproduce the long-memory property in the variance. Based on the diagnostic tests, we claim that the nonstationary TV-GARCH model should be preferred to the stationary model in applications using long financial data.

Moreover, the results indicate that the first-order GJR-GARCH model is inadequate to describe the short-run dynamics of volatility over long return series, and another type of nonlinear model should be considered. Further improvements in the modelling of the conditional variance over long time series are needed, but this problem is left for further research.
References


Appendix A: Figures

Figure 8: Estimated $g_t$ functions for the five subperiods.
Figure 9: Conditional standard deviations of the GJR-GARCH(1,1) and the TV-GJR-GARCH(1,1) model for the five subperiods.
<table>
<thead>
<tr>
<th>Subsample 1 (02/01/1920 – 31/12/1931): T=3004</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon_t$</td>
</tr>
<tr>
<td>$\varepsilon_t/\hat{g}_{1/2}$</td>
</tr>
</tbody>
</table>

Subsample 2 (04/01/1932 – 31/12/1943): T=2995

| $\varepsilon_t$ | -8.778 | 14.27 | 0.019 | 1.652 | 0.198 | 7.178 | -0.015 | 0.389 |
| $\varepsilon_t/\hat{g}_{1/2}$ | -4.785 | 5.824 | 0.011 | 0.860 | -0.173 | 3.684 | 0.008 | 0.266 |

Subsample 3 (04/01/1944 – 29/12/1961): T=4511

| $\varepsilon_t$ | -6.766 | 4.048 | 0.037 | 0.702 | -0.869 | 6.378 | 0.008 | 0.145 |
| $\varepsilon_t/\hat{g}_{1/2}$ | -6.766 | 4.048 | 0.036 | 0.660 | -0.818 | 6.054 | 0.013 | 0.143 |

Subsample 4 (01/01/1962 – 16/11/1982): T=5241

| $\varepsilon_t$ | -5.882 | 4.952 | 0.006 | 0.844 | 0.251 | 2.908 | -0.015 | 0.123 |
| $\varepsilon_t/\hat{g}_{1/2}$ | -5.878 | 4.576 | 0.005 | 0.656 | 0.185 | 3.822 | -0.002 | 0.068 |


| $\varepsilon_t$ | -25.63 | 9.666 | 0.047 | 1.087 | -4.768 | 115.57 | -0.020 | 0.187 |

Subsample 6 (03/01/1994 – 31/12/2003): T=2557

| $\varepsilon_t$ | -7.455 | 6.155 | 0.040 | 1.111 | -0.261 | 4.131 | 0.030 | 0.232 |
| $\varepsilon_t/\hat{g}_{1/2}$ | -5.655 | 4.262 | 0.033 | 0.814 | -0.287 | 3.653 | 0.040 | 0.148 |

Notes: The table contains summary statistics for each of the subperiod series. The sample periods are indicated in parentheses. The statistic ‘S.D.’ is the standard deviation, ‘Skew’ is the coefficient of skewness and the statistic ‘Ex.Kr’ is the value of the excess kurtosis. ‘Rob.Sk.’ denotes the robust measure for skewness and ‘Rob.Kr.’ denotes the robust centred coefficient for kurtosis. ‘Rob.Sk.’ is computed as $SK = (Q_3 + Q_1 - 2Q_2)/(Q_3 - Q_1)$ where $Q_i$ is the $i$th quartile of the returns and ‘Rob.Kr.’ is computed as $KR = (E_7 - E_5 + E_3 - E_1)/(E_6 - E_2) - 1.23$ where $E_i$ is the $i$th octile (see Kim and White (2004) for details).
Table 10: Estimation results of the TV-GJR-GARCH(1,1) model: subperiods

<table>
<thead>
<tr>
<th>Subsample</th>
<th>Equation</th>
<th>Log-Lik</th>
<th>g_t</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subsample 1</td>
<td>( h_t = 0.0668 + 0.0032 \varepsilon_{t-1}^2 + 0.8244h_{t-1} + 0.1575I_{t-1}(\varepsilon_{t-1}^* &lt; 0)\varepsilon_{t-1}^2 )</td>
<td>-3677.8</td>
<td>0.3738 + 0.4779(1 + exp{ -24.379(t^* - 0.0856)(t^* - 0.7228) })^{-1} + 2.3966(1 + exp{ -100(t^* - 0.8109) })^{-1}</td>
<td>( \hat{\alpha}_1 + \hat{\beta}_1 + \hat{\kappa}_1/2 = 0.9063 )</td>
</tr>
<tr>
<td>Subsample 2</td>
<td>( h_t = 0.0205 + 0.0338 \varepsilon_{t-1}^2 + 0.8876h_{t-1} + 0.1041I_{t-1}(\varepsilon_{t-1}^* &lt; 0)\varepsilon_{t-1}^2 )</td>
<td>-3524.3</td>
<td>1.1331 - 0.7717(1 + \exp{ -100(t^* - 0.2128) })^{-1} - 0.2130(1 + \exp{ -100(t^* - 0.7395) })^{-1}</td>
<td>( \hat{\alpha}_1 + \hat{\beta}_1 + \hat{\kappa}_1/2 = 0.9735 )</td>
</tr>
<tr>
<td>Subsample 3</td>
<td>( h_t = 0.0423 + 0.0007 \varepsilon_{t-1}^2 + 0.8331h_{t-1} + 0.1348I_{t-1}(\varepsilon_{t-1}^* &lt; 0)\varepsilon_{t-1}^2 )</td>
<td>-4298.5</td>
<td>1 - 0.2369(1 + \exp{ -100(t^* - 0.3937) })^{-1}</td>
<td>( \hat{\alpha}_1 + \hat{\beta}_1 + \hat{\kappa}_1/2 = 0.9012 )</td>
</tr>
<tr>
<td>Subsample 4</td>
<td>( h_t = 0.0076 + 0.0203 \varepsilon_{t-1}^2 + 0.9191h_{t-1} + 0.0884I_{t-1}(\varepsilon_{t-1}^* &lt; 0)\varepsilon_{t-1}^2 )</td>
<td>-4769.2</td>
<td>1 + 2.4727(1 + \exp{ -14.373(t^* - 0.5568) })^{-1} - 1.4488(1 + \exp{ -100(t^* - 0.6775) })^{-1}</td>
<td>( \hat{\alpha}_1 + \hat{\beta}_1 + \hat{\kappa}_1/2 = 0.9836 )</td>
</tr>
<tr>
<td>Subsample 5</td>
<td>( h_t = 0.0197 + 0.9026h_{t-1} + 0.1400I_{t-1}(\varepsilon_{t-1}^* &lt; 0)\varepsilon_{t-1}^2 )</td>
<td>-2883.1</td>
<td>1 + 1.0949(1 + \exp{ -8.3780(t^* - 0.2967) })^{-1}</td>
<td>( \hat{\alpha}_1 + \hat{\beta}_1 + \hat{\kappa}_1/2 = 0.9726 )</td>
</tr>
</tbody>
</table>

Notes: The table contains the parameter estimates from the TV-GJR(1,1) model for each of the subperiods of the DJIA daily returns from January 2, 1920 until December 31, 2003. The estimated model has the form of the equations (5)-(8). The numbers in parentheses are the standard errors.
Table 11: Estimation results of the GJR-GARCH(1,1) model: subperiods

<table>
<thead>
<tr>
<th>Subsample</th>
<th>( h_t )</th>
<th>Log-Lik</th>
<th>( \hat{\alpha}_1 + \hat{\beta}_1 + \hat{\kappa}_1/2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 0.0467 + 0.0230\varepsilon_{t-1}^2 + 0.8752h_{t-1} + 0.1399I_{t-1}(\varepsilon_{t-1}^* &lt; 0)\varepsilon_{t-1}^2 )</td>
<td>-4675.49</td>
<td>0.9680</td>
</tr>
<tr>
<td>2</td>
<td>( 0.0168 + 0.0349\varepsilon_{t-1}^2 + 0.9126h_{t-1} + 0.0930I_{t-1}(\varepsilon_{t-1}^* &lt; 0)\varepsilon_{t-1}^2 )</td>
<td>-4911.1</td>
<td>0.9940</td>
</tr>
<tr>
<td>3</td>
<td>( 0.0332 + 0.0015\varepsilon_{t-1}^2 + 0.8732h_{t-1} + 0.1111I_{t-1}(\varepsilon_{t-1}^* &lt; 0)\varepsilon_{t-1}^2 )</td>
<td>-4542.8</td>
<td>0.9302</td>
</tr>
<tr>
<td>4</td>
<td>( 0.0045 + 0.0252\varepsilon_{t-1}^2 + 0.9310h_{t-1} + 0.0814I_{t-1}(\varepsilon_{t-1}^* &lt; 0)\varepsilon_{t-1}^2 )</td>
<td>-5908.2</td>
<td>0.9969</td>
</tr>
<tr>
<td>5</td>
<td>( 0.0359 + 0.0279\varepsilon_{t-1}^2 + 0.8932h_{t-1} + 0.0941I_{t-1}(\varepsilon_{t-1}^* &lt; 0)\varepsilon_{t-1}^2 )</td>
<td>-3736.4</td>
<td>0.9681</td>
</tr>
<tr>
<td>6</td>
<td>( 0.0189 + 0.0059\varepsilon_{t-1}^2 + 0.9169h_{t-1} + 0.1299I_{t-1}(\varepsilon_{t-1}^* &lt; 0)\varepsilon_{t-1}^2 )</td>
<td>-3587.5</td>
<td>0.9878</td>
</tr>
</tbody>
</table>

Notes: The table contains the parameter estimates from the GJR(1,1) model for each of the subperiods of the DJIA daily returns from January 2, 1920 until December 31, 2003. The estimated model has the form \( h_t = \omega + \alpha_1\varepsilon_{t-1}^2 + \beta_1h_{t-1} + \kappa_1I_{t-1}(\varepsilon_{t-1} < 0)\varepsilon_{t-1}^2 \), where \( I_{it}(\varepsilon_{it}) = 1 \) if \( \varepsilon_{it} < 0 \) (and 0 otherwise) for all \( i \). The numbers in parentheses are the Bollerslev-Wooldridge robust standard errors.
Table 12: GPH estimates of the long-memory parameter

<table>
<thead>
<tr>
<th>Periods</th>
<th>$d_{GPH}(m = T^{0.4})$</th>
<th>$d_{GPH}(m = T^{0.5})$</th>
<th>$d_{GPH}(m = T^{0.6})$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\varepsilon_t$</td>
<td>$\varepsilon_t/\hat{g}_t^{1/2}$</td>
<td>$\varepsilon_t$</td>
</tr>
<tr>
<td>Subsample 1</td>
<td>0.3688</td>
<td>-0.171</td>
<td>0.4237</td>
</tr>
<tr>
<td></td>
<td>(0.1434)</td>
<td>(0.1214)</td>
<td>(0.0866)</td>
</tr>
<tr>
<td>Subsample 2</td>
<td>0.7285</td>
<td>0.2173</td>
<td>0.6442</td>
</tr>
<tr>
<td></td>
<td>(0.1199)</td>
<td>(0.1918)</td>
<td>(0.0889)</td>
</tr>
<tr>
<td>Subsample 3</td>
<td>0.2958</td>
<td>0.2454</td>
<td>0.3134</td>
</tr>
<tr>
<td></td>
<td>(0.1618)</td>
<td>(0.1493)</td>
<td>(0.0885)</td>
</tr>
<tr>
<td>Subsample 4</td>
<td>0.4457</td>
<td>0.2228</td>
<td>0.4728</td>
</tr>
<tr>
<td></td>
<td>(0.1330)</td>
<td>(0.1179)</td>
<td>(0.0893)</td>
</tr>
<tr>
<td>Subsample 5</td>
<td>0.2996</td>
<td>0.2996</td>
<td>0.4420</td>
</tr>
<tr>
<td></td>
<td>(0.0974)</td>
<td>(0.0974)</td>
<td>(0.0819)</td>
</tr>
<tr>
<td>Subsample 6</td>
<td>0.4776</td>
<td>0.3804</td>
<td>0.4250</td>
</tr>
<tr>
<td></td>
<td>(0.1543)</td>
<td>(0.1820)</td>
<td>(0.0924)</td>
</tr>
</tbody>
</table>

Notes: The numbers in parentheses are the standard errors. The bandwidth $m$ equals $T^{\alpha}, \alpha \in \{0.4, 0.5, 0.6\}$ where $T$ is the number of observations.